Solution of Pressure-Poisson Equation (PPE) in Irregular Domains with non-Staggered Grid Arrangements

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ABSTRACT

The Pressure-Poisson Equation (PPE) is a classical elliptic partial differential equation which provides a relationship to extract the so-called pressure parameter from a distributed parameter on an enclosed domain. The proper formulation for the PPE together with the boundary conditions must satisfy the compatibility and incompressibility conditions and minimize the resulting errors of approximations and discretization. Especially, when a non-staggered arrangement of parameters in an irregular domain is considered, satisfying these constraints needs more considerations. In this paper, solution of the PPE with the Neumann boundary condition on an irregular domain using this type of grid arrangement is considered. In this regard, the proper discretization of the PPE for this solution is presented. Also, a method of boundary and domain coding is suggested to facilitate applying boundary conditions at irregular domains. The convergency of this solution method is evaluated using a test problem. The results show the applicability and correct discretization approach provided in this study.

KEYWORDS

Numerical Solution, Pressure Poisson Equation, Irregular Domain, non-Staggered Grid.

1. INTRODUCTION

The Pressure Poisson Equation (PPE) is a standard elliptic partial differential equation which widely appears in the numerical formulation and solution of the incompressible Navier-Stokes equations using projection methods [1], [2]. It uses distributed velocity vector data to obtain the unknown potential parameter pressure. It is shown that the arrangement of these parameters in the computational domain plays an important role in discretization and solution of this equation [3]. Conventional methods apply the staggered arrangement of the contributing parameters in the solution of this equation [4]. But there is some applications in which the data is collocated in a single grid point and/or the unknown parameters are necessary to be calculated at the same data points. They show the necessity of solving the PPE using non-staggered grids. Implementation of this type of computational mesh provides several advantages [3], [5], but it suffers from oscillation in the calculated pressure distribution and the difficulties posed by the integral compatibility constraint to obtain a unique solution for the pressure [3]. Some publications have proposed corrections required for the numerical solution of the PPE at regular domains to satisfy the compatibility condition [6], [7]. But the formulation for irregular domains has not been developed yet. This paper suggests some strategies in the numerical solution of the PPE over irregular domains. The collocated grid arrangement is considered in this regard. The type of boundary condition, the form of governing equation, the method of domain and boundary discretization and organization of irregular domain grids for applying these conditions must be considered in these calculations.

Gresho and Sani (1987) investigated the problem of the pressure boundary conditions and found the Neumann boundary condition (normal momentum equation on the boundary) to be the appropriate one for the PPE. Both Neumann and Dirichlet boundary conditions provide a unique numerical solution for the PPE [8]. Numerical studies show the same result when these boundary conditions are used on the primitive and non-primitive formulation of the PPE [9].

The solution for the PPE with the Neumann boundary conditions is obtained only if a compatibility condition is satisfied. This condition relates the source of the PPE and the Neumann boundary conditions (Green's theorem).

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Failure to satisfy the compatibility condition leads to non-convergent iterative solutions [10]. This condition will be satisfied by selecting an appropriate technique for boundary and domain discretization. Finding the proper boundary condition for all types of boundary nodes is important particularly at irregular domains, since in combination with the domain equations it provides the compatibility conditions. An organization of domain nodes is necessary to prevent the complexity and the misalliance resulting from applying the governing equations to the obtained nodes of the region.

In this study, the solution of the PPE at irregular domains with a finite difference discretization scheme is considered. This study extends the previous studies for providing consistency in conditions for the numerical solution of the PPE on irregular domains. The required formulation and the necessary boundary conditions for this equation are presented. Also, to facilitate the numerical implementation of these techniques on irregular domains, an effective masking matrix is suggested. Using this masking tool, a boundary coding scheme is developed which improves the numerical solution of the resulting equations at irregular domains. This paper is organized as follows; first, the PPE is introduced and its proper discretization on irregular domains is presented. A method for effective organization of boundary and domain equations is provided in the next step. Finally, the applicability of these strategies is evaluated using a test problem and a discussion on the results is provided.

2. Materials and Methods

A. Governing Equations

The conventional Pressure Poisson equation is obtained from the primitive form of the Navier-Stokes equations. For the better understanding of the contributing parameters and their applications, we extract the PPE from the governing equations of an incompressible Newtonian fluid domain. In an incompressible inviscid flow domain, the continuity equation is as follows:

$$\nabla.\mathbf{v} = \mathbf{0} \tag{1}$$

where $\mathbf{v}$ is the velocity vector and $\nabla.$ is the divergence operator. The momentum equation is also as:

$$\mathbf{v}_t + (\mathbf{v}.\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu\Delta\mathbf{v} \tag{2}$$

where p is the static pressure, ρ the density, and ν is the kinematic viscosity, and Δ is the laplacian operator. Conventionally, in the solution of these equations, the momentum equation (2) clearly determines the respective velocity components. This leaves the continuity equation (1) to determine the pressure. When the velocity data in a domain is known, finding the related pressure distribution is performed by a special operation on this equation. The form of continuity equation suggests that we take the divergence of the momentum equation (2). The continuity equation (1) can then be used to simplify the resulting equation, leaving a Pressure-Poisson equation for the static pressure. The viscous and unsteady terms disappear by virtue of the continuity equation to obtain:

$$\Delta p = -\rho\nabla.(\mathbf{v}.\nabla)\mathbf{v} \tag{3}$$

The pressure equation (3) can be solved by one of the numerical methods for elliptic equations. In two-dimensional problems, the pressure equation (3) can be summarized as below:

$$p_{xx} + p_{yy} = -\rho[(u\tfrac{\partial u}{\partial x} + v\tfrac{\partial u}{\partial y})_x + (u\tfrac{\partial v}{\partial x} + v\tfrac{\partial v}{\partial y})_y] \tag{4}$$

where u and v are the velocity components and the subscripts x and y in all the governing equations refer to partial derivatives with respect to x and y.

B. Numerical Solution Procedures

In this section, a method is described to integrate numerically the two dimensional equations of the primitive PPE. The governing equation (3) must be discretized on a collocated grid. A well-posed boundary value problem, for the elliptic pressure equation requires the conditions on all the boundaries of the computational domain to be specified. In addition to the governing equation, the boundary conditions should also be derived. One approach is to set the momentum equation valid as well as on the boundary itself. However, this is a vector equation and only one scalar boundary condition is required. With the normal projection of the momentum equation (2) upon the boundary, the Neumann boundary condition for the static PPE is obtained as below:

$$p_{\mathbf{n}} = \mathbf{n}.\nabla p \tag{5}$$

Here, $\mathbf{n}$ is the outward normal to the boundary and subscript $\mathbf{n}$ is referring to partial derivatives with respect to this normal vector. Existence of a solution for the PPE with this Neumann boundary condition in both formulations requires the satisfaction of the compatibility constraint,

$$\iint_A \Delta p dA = \oint_S p_{\mathbf{n}} ds \tag{6}$$

The area of the solution domain, A, is enclosed by the boundary contour S. Equation (6) is a consequence of Green's theorem. Failure to satisfy this condition causes the iterative solution to drift slowly and endlessly [10]. It is shown that the compatibility condition (6) is not automatically satisfied on a non-staggered grid [6]. With reference to Figure 1 the two dimensional PPE (4) will be discretized for inner nodes such that to prevent the pressure oscillation on the domain and to use the collocated node data. The satisfaction of the compatibility condition is associated with proper discretization of the boundary equations. As stated before, the discretization will proceed first by discretization of the momentum equation and then the divergence operator will be applied.

For an irregular domain similar to Figure 1, the boundary nodes are classified at 8 kinds of boundary conditions. In the following, the appropriate Neumann boundary condition for each of these boundary types is developed to satisfy the compatibility condition (6). The discretization is performed by considering a unique grid size h in both x and y directions.

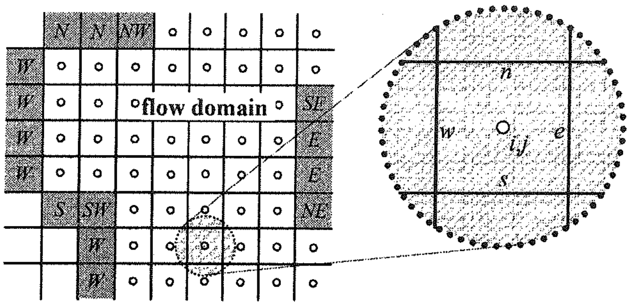


Figure 1: A schematic of a two dimensional arbitrary irregular flow domain, grid specifications and the classification of boundary nodes. The boundary nodes are gray and the domain nodes are indicated with white circles.

Equation (4) can be discretized as given below:

$$p_{i+1,j} + p_{i-1,j} + p_{i,j+1} + p_{i,j-1} - 4p_{i,j} = - \rho h[(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y})_e - (u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y})_w + (u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y})_n - (u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y})_s] \quad (7)$$

where

$$e_{i,j} : (u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y})_e = \frac{(u_{i+1,j}+u_{i,j})(u_{i+1,j}-u_{i,j})}{2h} + \frac{(v_{i+1,j}+v_{i,j})(u_{i+1,j+1}+u_{i,j+1}-u_{i+1,j-1}-u_{i,j-1})}{8h}$$

$$w_{i,j} : (u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y})_w = \frac{(u_{i,j}+u_{i-1,j})(u_{i,j}-u_{i-1,j})}{2h} + \frac{(v_{i,j}+v_{i-1,j})(u_{i,j+1}+u_{i-1,j+1}-u_{i,j-1}-u_{i-1,j-1})}{8h}$$

$$n_{i,j} : (u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y})_n = \frac{(u_{i,j+1}+u_{i,j})(v_{i+1,j+1}+v_{i+1,j}-v_{i-1,j+1}-v_{i-1,j})}{8h} + \frac{(v_{i,j+1}+v_{i,j})(v_{i,j+1}-v_{i,j})}{2h} \quad (8)$$

$$s_{i,j} : (u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y})_s = \frac{(u_{i,j}+u_{i,j-1})(v_{i+1,j-1}+v_{i+1,j}-v_{i-1,j-1}-v_{i-1,j})}{8h} + \frac{(v_{i,j}+v_{i,j-1})(v_{i,j}-v_{i,j-1})}{2h}$$

Boundary condition (5) can also discretized as follows,

$$E_{i,j} : -p_{i,j} + p_{i-1,j} = \rho h\left[+ w_{i,j} - \nu(\frac{u_{i,j}-2u_{i-1,j}+u_{i-2,j}}{h^2} + \frac{u_{i-1,j+1}-2u_{i-1,j}+u_{i-1,j-1}}{h^2})\right]$$

$$W_{i,j} : +p_{i+1,j} - p_{i,j} = \rho h\left[- e_{i,j} + \nu(\frac{u_{i,j}-2u_{i+1,j}+u_{i+2,j}}{h^2} + \frac{u_{i+1,j+1}-2u_{i+1,j}+u_{i+1,j-1}}{h^2})\right]$$

$$N_{i,j} : -p_{i,j} + p_{i,j-1} = \rho h\left[+ s_{i,j} - \nu(\frac{v_{i+1,j-1}-2v_{i,j-1}+v_{i-1,j-1}}{h^2} + \frac{v_{i,j}-2v_{i,j-1}+v_{i,j-2}}{h^2})\right]$$

$$S_{i,j} : +p_{i,j+1} - p_{i,j} = \rho h\left[- n_{i,j} + \nu(\frac{v_{i+1,j+1}-2v_{i,j+1}+v_{i-1,j+1}}{h^2} + \frac{v_{i,j}-2v_{i,j+1}+v_{i,j+2}}{h^2})\right] \quad (9)$$

$$NE_{i,j} : +p_{i-1,j} + p_{i,j-1} - 2p_{i,j} = N_{i,j} + E_{i,j}$$

$$NW_{i,j} : +p_{i+1,j} + p_{i,j-1} - 2p_{i,j} = N_{i,j} + W_{i,j}$$

$$SE_{i,j} : +p_{i-1,j} + p_{i,j+1} - 2p_{i,j} = S_{i,j} + E_{i,j}$$

$$SW_{i,j} : +p_{i+1,j} + p_{i,j+1} - 2p_{i,j} = S_{i,j} + W_{i,j}$$

Here, the negative sign of the boundary conditions is considered. In this case, for the evaluation of the compatibility equation (6), both sides of equation (7) on all nodes of the whole domain will be summed with that of equation (9) on all boundary nodes. For discretization of the viscous term, as presented in the right hand side of (9), it is assumed that the second derivative of velocity components on the boundary nodes and its adjacent nodes are equal.

For providing the appropriate Neumann boundary condition at the edge nodes (presented with *NE*, *SE*, *NW* and *SW*), it is assumed that the domain nodes in both sides of these edge nodes provide equal contribution on its pressure value. Using this type of boundary condition, the consistency of pressure members on the left hand side, and the convective terms on the right hand side of equations (7) and (9) will vanish, and only the viscous terms from the boundary conditions remain on the right hand side. To check the dissipation of this term, from the Green's theorem for a given vector function $\Delta \mathbf{v}$ we have:

$$\oint_S \Delta \mathbf{v}.\mathbf{n} ds = \iint_A \nabla.\Delta \mathbf{v}\, dA \quad (10)$$

From the continuity equation, the term on the right hand side of this integral equation will vanish; hence we see that for equal size boundary elements ds, the summation of $\Delta \mathrm{v}$ projection along the normal over the whole boundary will vanish. This states that the summation of the viscous terms on the right hand side of the consistency equation over the boundary is equal to zero; therefore, with this type of discretization and boundary conditions the compatibility constraint will be satisfied.

C. Construction of Mask Matrix

Mask matrix is a tool for providing ease in numerical implementation of such a system of equations on an irregular domain. Utilizing this matrix, a method arises for arrangement and coding of the grids in the domain. The index which specifies the computational node data is the same as the index implemented for specifying the code of the nodes in the mask matrix. Figure 2 provides an illustration of the mask matrix components for a sample irregular domain. All nodes inside the domain take a similar value at the mask matrix. A different value is considered for all outside nodes. Through this, it is possible to find out immediately, while programming, whether a node is inside, outside, or on the boundary. For example in Figure 2 these values are set to *1* and *0* for inside and outside nodes respectively. Also, using this mask matrix, a boundary coding approach was adopted for specifying each kind of boundary node. Each boundary node, which according to its role in the boundary are presented with *E*, *W*, *N*, *S*, *NE*, *NW*, *SE*, and *SW*, adopts a value for example between *10* to *80*, respectively. As a result, during the iterative solution, all nodes in the numerical domain can be scanned and according to their code at the mask matrix, the related appropriate equation is considered.

0	30	30	60	1	1	1	1	1
20	1	1	1	1	1	1	1	1
20	1	1	1	1	1	1	1	70
20	1	1	1	1	1	1	1	10
20	1	1	1	1	1	1	1	10
0	40	80	1	1	1	1	1	50
0	0	20	1	1	1	1	1	1
0	0	20	1	1	1	1	1	1

Figure 2: A schematic of mask matrix components according to their position in the domain. Here, inside of the domain is coded with *1*, outside with zero and boundary nodes are coded from *10* to *80* according to their types.

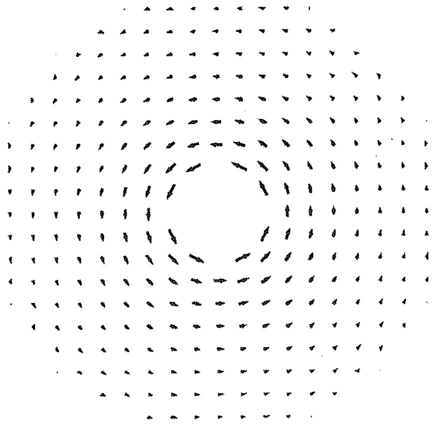


Figure 3: The Couette flow velocity vector field.

3. Results and Discussion

Numerical evaluation of the primitive form of the PPE is provided with a test problem in which the real velocity domain and the related pressure distribution are available. Here, a standard Couette flow domain is considered for this evaluation. The Couette flow system consists of two co-centric cylinders with fluid filled in the annular region. The relative rotation of these two cylinders around their axes of symmetry provides a rotational flow domain. The Navier-Stokes equation for this particular geometry can be solved in cylindrical coordinates [11]. It can be shown that the velocity field of fluid is as below,

$$v = c_1 r + c_2/r \tag{11}$$

and the pressure is as;

$$p = \rho(c_1^2 r^2 - c_2^2/r^2 + 4c_1 c_2 \ln r) + c_3 \tag{12}$$

where c_1 and c_2 are constants which are found from the boundary conditions and c_3 is the integration constant. Now, we are equipped with a nontrivial divergence-free velocity field with a known pressure distribution. Figure 3 shows the velocity domain provided from (11).

The successive over-relaxation (SOR) iterative method is employed in the numerical estimation of the pressure domain. For the primitive form of the PPE, equation (7) with boundary condition (9) is solved. A schematic of the components of mask matrix constructed for this Couette flow system is presented in Figure 4. Since the domain nodes accept a similar discretized equation, they are presented with a unique coding value. The outside nodes have zero impact on the calculations and those which have coded similarly will be rejected during the run. Other values in this figure are distinctive of the kinds of boundary nodes.

Numerical solution for the Couette flow domain is obtained using an equal grid size h in both x and y directions. The calculated pressure contours from this primitive formulation of PPE are compared with the standard distribution according to Figure 5. It is clear that the result presented from the standard pressure distribution is repeated with this method. These results are obtained after 100 iterations. Smoothness and similarity of the obtained pressure contours in Figure 5 (b) indicate that the adopted boundary condition is correct and well-founded.

Better understanding of the effect of calculation of static pressure distribution can be obtained with comparison of the pressure distribution on the centerline of the Couette flow domain. Figure 6 shows the distribution obtained from the PPE after 100 iterations. According to this figure, there is a good agreement between the results and the actual pressure distribution.

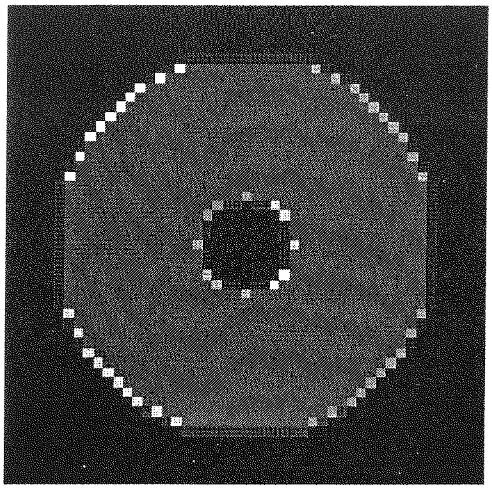
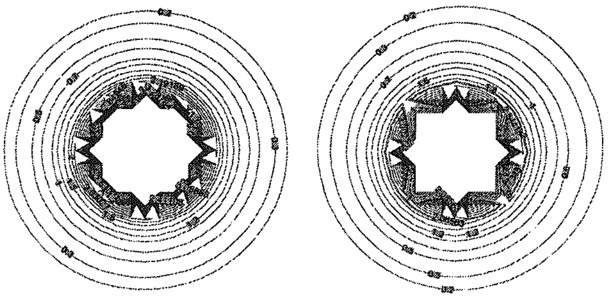


Figure 4: A presentation of mask matrix in considered model. The continuous gray presents the inside flow region, dark pixels are the outside points and the remaining are the coded boundary nodes.



(a) (b)

Figure 5: Comparison of (a) the real pressure contours and (b) the pressure distribution obtained from static PPE formulation.

It is evident from (1) and (2) that the momentum and continuity equations obtain only the values of the pressure gradient, hence the calculation of pressure values from these equations may be somewhat offset. Considering this offset, in order to obtain comparable curves in Figure 6, the mean value of whole pressure distribution obtained from the primitive formulation of the PPE is set equal to that of the standard pressure distribution. This is the main reason for the different zero regions outside of the flow domain (at location greater than +1, lower than -1 and between -0.2 and 0.2).

To be confident of the pressure distribution results and to find the number of iterations required for obtaining the most precise estimation, the estimated pressure is compared with the standard pressure distribution by calculating a normalized error. The mean normalized error considered for this comparison is defined as follows:

$$\textbf{Mean Normalized Error} = \frac{1}{N}\sum_{i=1}^{N}\frac{|p_i - \hat{p}_i|}{p_{\max} - p_{\min}} \qquad (13)$$

Here, $\hat{p}_i$ is the calculated pressure, p_i is the real pressure, $p_{\min}$ is the minimum and $p_{\max}$ is the maximum real pressure in the domain.

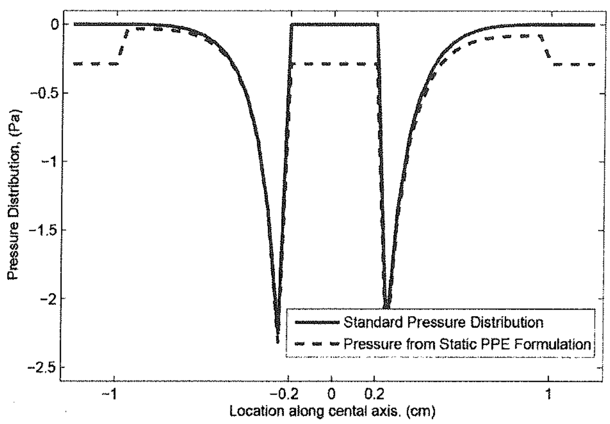


Figure 6: Pressure distribution obtained from primitive (dashed line) formulation of PPE and its comparison with the real pressure (solid line) along the centerline of Couette flow domain.

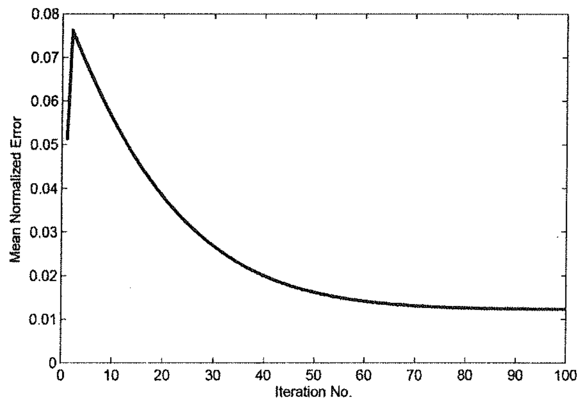


Figure 7: Mean Normalized Error obtained from the static pressure formulation of PPE.

Again, because of the offset in the calculated pressure, first, the mean value of the whole domain pressure distribution from these formulations is set equal to that of the standard one. Finally, the obtained $\hat{p}_i$ from this correction is used in equation (13). Figure 7 shows this normalized error as a function of iteration number for 100 iterations. This figure shows that the pressure distribution obtained from the primitive variable formulation provides correct pressure distribution. On the other hand, the primitive PPE is unstable up to 10 iterations and after 60 iterations, the solution converges to the actual result. Performing more iteration will provide better results from these methods but would not change the order of convergence. Repeating these experiments with smaller grid size h shows similar results. It shows that in large grid sizes the primitive formulation is stable.

4. Conclusion

In this paper, a solution for the Pressure-Poisson Equation (PPE) on an irregular domain is presented. The

Neumann boundary condition is adopted for the solution. The primitive variable formulation for the extraction of the Pressure-Poisson Equation is developed. The challenging issue addressed in this study is using non-staggered grid arrangement in the flow domain. Satisfaction of the compatibility condition and oscillation of resulting pressure domain are the shortcomings of the solution of the PPE on this type of grid arrangements. An appropriate discretization method for the PPE and its appropriate boundary condition is developed. A specific kind of discretization is considered to prevent the pressure oscillation by sharing data with the adjacent grids. The necessary boundary nodes for discretization are specified and then classified in specific groups. The application of appropriate boundary condition for the eight types of boundary nodes together with the domain discretization, leads to the satisfaction of the compatibility condition. A specific mask matrix and boundary coding approach is used to facilitate the application of the appropriate equation on the irregular domain. The presented method is applied on a standard test problem of an irregular flow domain. It is shown that the actual pressure distribution is achieved by applying the described scheme.

5. References

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