Time-Optimal Switching Control Computation of Nonlinear Systems Using LGI Method

M. R. Navabi[i], M. Shamsi[ii] and M. Mani[iii]

ABSTRACT

An accurate and efficient computational method is proposed for the time-optimal switching control problem of nonlinear systems. The method is based on spectral collocation technique using the Legendre-Gauss nodes. Spectral method is used to discretize the problem in a sequence of time subdomains separately and form a nonlinear programming problem. By solving this NLP problem, the optimal control in each subdomain and the joining times of the subdomains will be obtained simultaneously. The method is computationally attractive and applications are demonstrated through illustrative examples.

KEYWORDS

time-optimal control; switching control; spectral; nonlinear system.

1. INTRODUCTION

The time-optimal control problem has been of great interest for several decades and has been used for a wide variety of applications such as flight trajectory optimization, industrial robotics and biomedical. In practical applications, the control function should be constrained to upper and lower bounds. When the system is linear in the control input and the control is bounded, the nonsingular time-optimal control solution is known theoretically to be the bang-bang control. Even, when the system is nonlinear in the control input, the most frequently encountered optimal control is bang-bang. The bang-bang control is a special case of piecewise-constant control that is referred to as switching control.

In general, time optimal switching control solutions can be hardly obtained analytically, unless the system is of low order, time-invariant and linear [1]. However, one can obtain the solution either by numerical solution of the two-point boundary value problem (TPBVP) arising from Pontryagin's minimum principle [2] or by numerical solution of mathematical programming problem that is formed by discretizing the optimal control problem [3]. The former approach is known as the indirect method and the later as the direct method.

Several algorithms have been proposed in the literature based on these two approaches to solve time-optimal switching control problem. Examples include the *Switching Time Variation Method* (STVM) presented by Mohler [4],[5], the *Switch Time Optimization* method (STO) by Meier and Brayson [6], the *Switching Time Computation* (STC) method and the *Time Optimal Switchings* (TOS) algorithm by Kaya and Noakes [7],[8], the *Control Parameterization Enhancing Technique* (CPET) by Lee et al [9], the *enhanced transcription scheme* by Hu et al [10], a shooting method by Bainum and Li [11], a smoothing technique by Bertrand and Epenoy [12], and a control transcription technique by Buskens et al [13].

The Spectral collocation methods have been recently applied to solve optimal control problems [14]-[15]. The spectral collocation methods are direct methods, which approximate the state and control functions with *N*th order Lagrange interpolating polynomials. The interpolating coefficients are the values of the function in collocation nodes. In this method, the dynamical system is discretized and the integral terms in the cost function is computed by a quadrature. Thereby, the optimal control problem is converted to a nonlinear programming (NLP) problem that can be solved by well-developed NLP solvers [16].

The spectral methods as direct methods are robust to the initial guess. They are easy to implement and can be applied to different problems without any extra work to derive the optimality necessary conditions for each

[i] Corresponding Author, Ph.D. Candidate, Amirkabir University of Technology (e-mail: navabi@aut.ac.ir).
[ii] Assistant Professor, Department of Applied Mathematics, Faculty of Computer Science and Mathematics, Amirkabir University of Technology, Tehran, Iran.
[iii] Professor, Department of Aerospace Engineering, Center of Excellence in Computational Aerospace Engineering, Amirkabir University of Technology, Tehran, Iran.

problem. Besides, the underlying approximation of the spectral method is of *spectral accuracy* for smooth problems. This means that the spectral method can offer an exponentially convergence rate [17].

However, the general methods are not appropriate for solving time optimal control problems where, there are discontinuities in the control function. The reason is that the switching points cannot be handled by spectral methods and spectral accuracy is lost for problems dealing with discontinuous functions. In this case, the Gibbs phenomenon appears in the solution [18], due to approximations of the discontinuous functions with global interpolating polynomials.

In this paper, to overcome the above difficulties, we propose a multidomain method to solve time-optimal switching control problem. In the proposed multidomain approach, the total domain is broken into several subdomains. The control and state functions are considered continuous in each subdomain and the switches can occur in the joining points. The spectral method is then applied to each subdomain. Using this approach, the resulted NLP problem includes also the switching times as unknown parameters.

In the proposed method, Integral Operation Matrix is used to discretize dynamical system and Legendre-Gauss nodes is used as nodal point. Therefore, this method is called Legendre-Gauss Integral (LGI) method throughout this paper.

Since the integral form is used, the initial boundary conditions are directly imposed to the equations. The Legendre-Gauss quadrature is used to apply final boundary conditions. This quadrature is exact for polynomials of degrees $2N-1$ and less, which is two degrees higher than Gauss-Lobatto quadratures with the same Lagrange interpolating polynomial order (N) [19].

To solve the time-optimal switching control problem, the Legendre-Gauss Integral (LGI) and Multidomain Legendre-Gauss Integral (MLGI) methods may be employed in a cascade way. At first, by using the Legendre-Gauss Integral method, a guess about the structure of control function and number of switches are obtained. Then, by applying MLGI method, optimal switching times, final time and bang-bang control function are accurately obtained.

This method provides several advantages over traditional spectral methods. The size of the resulted mathematical programming problem is smaller than others. Also, the Gibbs phenomena does not appear in the resulted solution and the *spectral accuracy* is preserved. Finally, the method can accurately obtain optimal switching times.

In the following section, the time optimal control problem is formulated in its differential form and then is converted to the integral form. We present the details of the Legendre-Gauss Integral method in section 3. In section 4, the multidomain Legendre-Gauss Integral method is described for time optimal switching control problem. Finally, in section 5, we use two numerical examples to illustrate our method and make a comparison with some of the results in the literature. The time-optimal control of Van der Pol and F-8 aircraft problems are studied. The results are in excellent agreement with those reported in the literature.

2. Problem Statement

Consider the time-optimal control problem

$$\min \quad \tau_f$$

$$\text{s. t.} \quad \dot{\mathbf{x}}(\tau) = \mathbf{f}(\mathbf{x}(\tau), \mathbf{u}(\tau), \tau) \qquad (1)$$

$$\mathbf{x}(0) = \mathbf{x}_0 \qquad (2)$$

$$\mathbf{x}(\tau_f) = \mathbf{x}_f \qquad (3)$$

$$\mathbf{u}_{\min} \le \mathbf{u}(\tau) \le \mathbf{u}_{\max} \qquad (4)$$

where $\mathbf{x}(\tau) \in \mathbb{R}^{n_x}$ is the state vector, $\mathbf{u}(\tau) \in \mathbb{R}^{n_u}$ is the control vector, τ is the time independent scalar variable and $\mathbf{f}: \mathbb{R}^{n_x} \times \mathbb{R}^{x_u} \times \mathbb{R} \to \mathbb{R}^{n_x}$ is a nonlinear vector function, considered to be continuously differentiable with respect to all its arguments. τ_f denotes final time, $\mathbf{x}_0$ and $\mathbf{x}_f$ denote initial and final states respectively, and $\mathbf{u}_{\min}$ and $\mathbf{u}_{\max}$ denote lower and upper bounds of control function. In other words, the goal is to find the history of control function $\mathbf{u}(\tau)$ and corresponding state trajectory $\mathbf{x}(\tau)$ that minimize the final time τ_f subject to the constraints imposed on the problem. The constraints include the dynamical constraints (1), boundary constraints as end conditions (2)-(3) and box constraints (4).

It is assumed that problem has an optimal solution. In this paper, the following integral form will be used instead of above equations

$$\min \quad \tau_f$$

$$\text{s.t.} \quad \mathbf{x}(\tau) = \mathbf{x}(0) + \int_0^{\tau} \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)\,dt \qquad (5)$$

$$\mathbf{x}(0) = \mathbf{x}_0 \qquad (6)$$

$$\mathbf{x}(\tau_f) = \mathbf{x}_f \qquad (7)$$

$$\mathbf{u}_{\min} \le \mathbf{u}(\tau) \le \mathbf{u}_{\max} \qquad (8)$$

3. Legendre-Gauss Integral (LGI) Method

A. *Preliminary considerations[18]*

Let $L_N(t)$, $-1 < t < 1$ denote the Legendre polynomial of order N. The Legendre-Gauss (LG) nodes $t_1, ..., t_N$, are the zeros of $L_N(t)$. No explicit formulas are known for LG nodes. However, they can be computed numerically [18].

Let $\phi_j(t)$, $j = 1, 2, ..., N$ be the Lagrange polynomials based on LG nodes, that are expressed as:

$$\phi_j(t)=\prod_{\substack{i=1\\ i\neq j}}^{N}\frac{t-t_i}{t_j-t_i},\qquad j=1,2,...,N \tag{9}$$

with the Kronecker property

$$\phi_j(t_i)=\delta_{ji}=\begin{cases}0 & j\neq i\\ 1 & j=i\end{cases} \tag{10}$$

It is more convenient to consider the alternative expression [19]

$$\phi_j(t)=\frac{1}{N(N+1)L_N(t_j)}\frac{(t^2-1)\dot{L}_N(t)}{(t-t_j)},\ j=1,2,...,N \tag{11}$$

The N th degree interpolation polynomial $h^N(t)$, to $h(t)$ is given by

$$h^N(t)=\sum_{j=1}^{N}h(t_j)\phi_j(t) \tag{12}$$

and therefore, by (10)

$$h^N(t_j)=h(t_j),\ \ j=1,2,...,N \tag{13}$$

the Legendre-Gauss quadrature rule can be used to approximate integral of a function over $[-1,1]$ as

$$\int_{-1}^{1}h(t)dt\approx\sum_{j=1}^{N}w_j h(t_j) \tag{14}$$

where w_j denotes the Legendre-Gauss weights [18] and can be computed as

$$w_j=\frac{2}{(1-t_j^2)}\frac{1}{[\dot{L}_{N+1}(t_j)]^2},\ j=1,2,...,N \tag{15}$$

where $\dot{L}_{N+1}(\cdot)$ denotes the derivative of $L_{N+1}(\cdot)$. The integral of a function over $[-1,t_j]$ can be approximated by

$$\int_{-1}^{t_j}h(t)dt\approx\sum_{k=1}^{N}S_{j,k}h(t_j) \tag{16}$$

where $S_{j,k}$ is the j th row and k th column component of $N\times N$ integration operational matrix $\mathbf{S}$. This matrix can be computed using Legendre-Gauss quadrature (14) for $[-1,t_j]$

$$\begin{aligned}S_{j,k}&=\int_{-1}^{t_j}\phi_k(t)dt\\ &=\frac{t_j+1}{2}\int_{-1}^{1}\phi_k\left(\left(\frac{t_j+1}{2}\right)(t+1)\right)dt\\ &=\frac{t_j+1}{2}\sum_{l=1}^{M}\phi_k(t_l)w_l\end{aligned} \tag{17}$$

B. Discretization of the optimal control problem

Consider the optimal control problem described by equations (5)-(8). Since the problem is formulated over the time interval $[0,\tau_f]$, and the Legendre Gauss nodes lie in the domain $[-1,1]$, we use the following linear transformation to map the physical domain to the computational domain $\tau\in[0,\tau_f]$:

$$\tau=\frac{\tau_f}{2}(t+1) \tag{18}$$

By applying the above transformation to equations (5)-(8), $\mathbf{x}$, $\mathbf{u}$ and $\mathbf{f}$ will change and we should use new symbols for them. However, for simplicity, we will use the same symbols. Therefore, equations (5)-(8) are reformulated as:

$$\min\ \ \tau_f$$

$$\text{s. t.}\quad \mathbf{x}(t)=\mathbf{x}(-1)+\frac{\tau_f}{2}\int_{-1}^{t}\mathbf{f}(\mathbf{x}(\tau),\mathbf{u}(\tau),\tau)d\tau \tag{19}$$

$$\mathbf{x}(-1)=\mathbf{x}_0 \tag{20}$$

$$\mathbf{x}(1)=\mathbf{x}_f \tag{21}$$

$$\mathbf{u}_{\min}\le\mathbf{u}(t)\le\mathbf{u}_{\max} \tag{22}$$

The Nth degree interpolating polynomials to $\mathbf{x}(t)=[x_1(t),...,x_{n_x}(t)]$ and $\mathbf{u}(t)=[u_1(t),...,u_{n_u}(t)]$ are expressed as

$$x_i^N(t)=\sum_{j=1}^{N}X_{i,j}\phi_j(t),\quad i=1,2,...,n_x \tag{23}$$

$$u_k^N(t)=\sum_{j=1}^{N}U_{k,j}\phi_j(t),\quad k=1,2,...,n_u \tag{24}$$

Note that from (13) we have

$$X_{i,j}=x_i^N(t_j),\ i=1,2,...,n_x,\ j=1,2,...,N \tag{25}$$

$$U_{k,j}=u_k^N(t_j),\ \ k=1,2,...,n_u,\ j=1,2,...,N \tag{26}$$

The above expansions can be expressed in the following matrix form,

$$\mathbf{x}^N(t)=\mathbf{X}\boldsymbol{\varphi}(t) \tag{27}$$

$$\mathbf{u}^N(t)=\mathbf{U}\boldsymbol{\varphi}(t) \tag{28}$$

where

$$\boldsymbol{\varphi}=[\phi_1(t)\ \ \phi_2(t)\ \ \cdots\ \ \phi_N(t)]^T \tag{29}$$

$\mathbf{X}$, $\mathbf{U}$ are $n_x\times N$ and $n_u\times N$ matrices with entries,

$$\mathbf{X}=[X_{i,j}],\quad i=1,2,...,n_x,\ j=1,2,...,N \tag{30}$$

$$\mathbf{U}=[U_{k,j}],\quad k=1,2,...,n_u,\ j=1,2,...,N \tag{31}$$

By using approximations (27) and (28), we can approximate $\mathbf{f}$ as

$$\begin{aligned}\mathbf{f}(\mathbf{x}(t),\mathbf{u}(t),t)&\approx\mathbf{f}\left(\mathbf{x}^N(t),\mathbf{u}^N(t),t\right)\\ &\approx\mathbf{F}\boldsymbol{\varphi}(t)\end{aligned} \tag{32}$$

where $\mathbf{F}$ is a $n_x\times N$ matrix

$$\mathbf{F}=[F_{i,j}], \tag{33}$$

$$F_{i,j} = \mathbf{f}_i(\mathbf{x}(t_j), \mathbf{u}(t_j), t_j), i = 1,2,...,n_x \ , \ j = 1,2,...,N \quad (34)$$

where $\mathbf{f}_i$ is the i th component of vector function $\mathbf{f}$.

By substituting (27), (28) and (32) in (19) and using integration operation matrix $\mathbf{S}$ we get to

$$\mathbf{X}\boldsymbol{\varphi}(t) = \mathbf{x}(-1) + \frac{\tau_f}{2}\mathbf{F}\boldsymbol{\varphi}(t)\mathbf{S}^T \quad (35)$$

By collocating the above equation in $t_j, j = 1,2,...,N$, finally we get to the following discretization of (19) in matrix form

$$\mathbf{X} = \mathbf{X}_0 + \frac{\tau_f}{2}\mathbf{F}\mathbf{S}^T \quad (36)$$

where $\mathbf{X}_0$ is a $n_x \times N$ matrix with same columns equal to $\mathbf{x}_0$:

$$\mathbf{X}_0 = [\mathbf{x}_0, ..., \mathbf{x}_0] \quad (37)$$

Using Gauss quadrature (14), we can compute the state vector at the final time:

$$\mathbf{x}^N(1) = \mathbf{x}_0 + \frac{\tau_f}{2}\mathbf{F}\mathbf{w} \quad (38)$$

where $\mathbf{w}$ is the vector of Legendre-Gauss quadrature weights

$$\mathbf{w} = [w_1, w_2, \cdots, w_N]^T \quad (39)$$

So the terminal condition (7), is disceretized as

$$\mathbf{x}_0 + \frac{t_f}{2}\mathbf{F}\mathbf{w} = \mathbf{x}_f \quad (40)$$

Finally, the optimal control problem is approximated by the following nonlinear optimization problem:

Find matrices $\mathbf{X}, \mathbf{U}$ and the final time τ_f to minimize

$J = \tau_f$

subject to

$$\mathbf{X} - \mathbf{X}_0 - \frac{\tau_f}{2}\mathbf{F}\mathbf{S}^T = 0 \quad (41)$$

$$\mathbf{x}_0 + \frac{\tau_f}{2}\mathbf{F}\mathbf{w} - \mathbf{x}_f = 0 \quad (42)$$

$$\mathbf{u}_{\min} \le \mathbf{u}^N \le \mathbf{u}_{\max} \quad (43)$$

4. MULTIDOMAIN LEGENDRE-GAUSS INTEGRAL (MLGI) METHOD:

The total domain $[\tau_0, \tau_f]$ is broken into n_{sd} subdomains and LGI method is applied to each subdomain. Let τ_s^k, be the switch times. Note that $n_{sd} = n_s + 1$.

where n_s is number of switching. Also, for simplicity in notation we set $\tau_s^0 = 0$ and $\tau_s^{n_s+1} = \tau_f$. So the considered subdomains are

$$[\tau_s^{k-1}, \tau_s^k], \ k = 1,2,..,n_s + 1 \quad (44)$$

Since, we are going to solve the bang-bang control problem, $\mathbf{u}(t)$ is considered constant in each subdomain. Let $\mathbf{u}(t)$ and $\mathbf{x}(t)$ in k th subdomain denoted by constant vector $\mathbf{u}^k$ and vector function $\mathbf{x}^k(t)$, respectively. In each subdomain the dynamic equations, are considered as

$$\mathbf{x}^k(\tau) = \mathbf{x}(0) + \int_{\tau_s^{k-1}}^{\tau} \mathbf{f}(\mathbf{x}^k(t), \mathbf{u}^k(t), t)dt \ , \ \tau_s^{k-1} \le t \le \tau_s^k \quad (45)$$

For imposing the continuity of the states at switch times, we consider $\mathbf{x}^k(0)$ in above equation as:

$$\mathbf{x}^k(0) = \begin{cases} \mathbf{x}^{k-1}(\tau_s^{k-1}) & , \ k = 2,...,n_s + 1 \\ \mathbf{x}_0 & , \ k = 1 \end{cases} \quad (46)$$

In a similar manner explained in the last section, we discretize the equation (45), but in this time, we consider $\mathbf{u}^k(t)$ to be constant in each subdomain

The final discretized form is

$$\mathbf{X}^k = \mathbf{X}_0^k + \frac{\tau_s^k - \tau_s^{k-1}}{2}\mathbf{F}^k(\mathbf{S}^k)^T, \ k = 1,...,(n_s + 1) \quad (47)$$

where $\mathbf{X}^k$ is a $n_x \times N_k$ coefficient matrix and $\mathbf{F}^k$ is a $n_x \times N_k$ matrix with entries,

$$F_{i,j}^k = f_i(\mathbf{x}^k(t_j), \mathbf{u}^k, t_j^k) \quad (48)$$

Also from (46), $\mathbf{X}_0^k$ is a $n_x \times N_k$ matrix that for $k = 1$, each of its columns is equal to $\mathbf{x}_0$ and for $k = 2,...,n_s + 1$ it is obtained by

$$\mathbf{x}_0^k = \mathbf{x}_0^{k-1} + \frac{\tau_s^k - \tau_s^{k-1}}{2}\mathbf{F}^{k-1}\mathbf{w}^{k-1} \quad (49)$$

The final condition (7) is applied to $\mathbf{X}^{n_s+1}$, as

$$\mathbf{x}_f = \mathbf{x}_0^{n_s} + \frac{\tau_s^{n_s} - \tau_s^{n_s-1}}{2}\mathbf{F}^{n_s}\mathbf{w}^{n_s} \quad (50)$$

Finally, the time-optimal bang-bang control problem (5)-(8) is converted to the following mathematical programming problem with unknown matrices $\mathbf{X}^k$, vectors $\mathbf{u}^k$, switch times τ_s^k, $k = 1,...,n_s + 1$ and final time τ_f,

$$\min \quad \tau_f$$

$$\mathbf{X}^k - \mathbf{X}_0^k - \frac{\tau_s^k - \tau_s^{k-1}}{2}\mathbf{F}^k(\mathbf{S}^k)^T = 0 \quad (51)$$

$$\mathbf{x}_0^{n_s} + \frac{\tau_s^{n_s} - \tau_s^{n_s-1}}{2}\mathbf{F}^{n_s}\mathbf{w}^{n_s} - \mathbf{x}_f = 0 \quad (52)$$

$$\mathbf{u}_{\min} \le \mathbf{u}^k \le \mathbf{u}_{\max} \quad (53)$$

5. ILLUSTRATIVE EXAMPLES

In this section, we use numerical examples to illustrate our method and make a comparison with some of the results in the literature. The Van der Pol problem and a model of F-8 aircraft will be studied.

The LGI and MLGI methods were implemented in MATLAB on a Pentium 4, 2GHz PC. The resulted nonlinear optimization problem may be solved by well developed optimization algorithms [16].

The LGI method provides an approximation of control function structure and final time. The result of this method, then, may be used as the initial guess for solving the nonlinear optimization problem resulted from MLGI method.

A. Time-optimal control of the Van der Pol problem

The nonlinear controlled Van der Pol equation is given as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - (x_1 - 1)x_2 + u \end{aligned} \tag{54}$$

Several researchers have treated the above equation with different boundary conditions and performance indexes. Here the time-optimal control of the Van der Pol problem is considered. Recently, Kaya and Noakes [7]-[8], Maurer and Osmolovskii [20] and Burachik et al [21] have treated this problem.

We use the following boundary conditions and constraint that has been taken from [20]

$\mathbf{x}_0 = [-0.40, 0.60]^T$ and $\mathbf{x}_f = [0.60, 0.40]^T$

and the following box constraint over control function

$-1 \le u \le 1$

The LGI method was employed to solve this problem. The control function history obtained by the LGI method for N =30 is shown in Figure 1. In this figure, the Gibbs phenomenon is observed which presents the discontinuity in the control function. This was expected, since the problem has discontinuity in its control function and the standard spectral methods are not appropriate to solve it. As it is known from theory, the optimal solution of this problem is a bang-bang control. The result in Figure 1 is not bang-bang control, but it shows the switching structure of the bang-bang control function and an approximation of switching time.

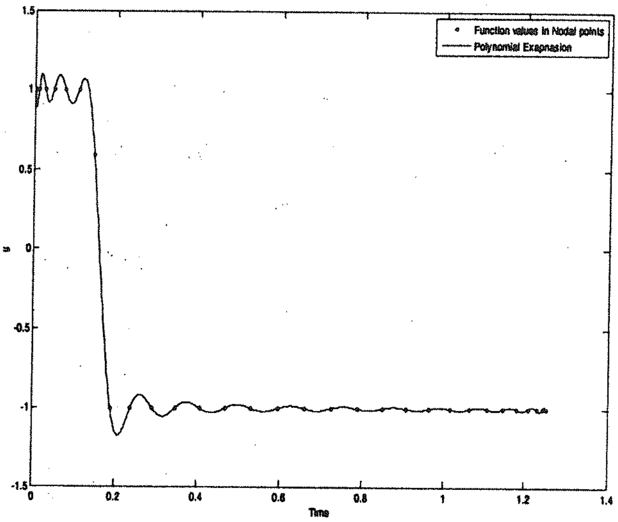

Figure 1: Control history of LGI method for Van der Pol problem, N =30, the dot symbols are the control function values in the LG nodes resulted by solving NLP problem. The solid line is the corresponding Lagrange expansion ($u^N(t)$).

To capture the optimal switching time and find the optimal final time, the MLGI method was employed. The results for various values of N are reported in Table 1.

TABLE 1

MLGI METHOD RESULTS FOR VAN DER POL PROBLEM FOR VARIOUS VALUES OF N

N	τ_s	τ_f	$\|E\|$
5	**0.15832008477693**	**1.25407514033795**	6.8604E-07
6	**0.15832014575317**	**1.25407473092406**	1.2299E-08
7	**0.15832014223252**	**1.25407472937036**	2.2563E-10
8	**0.15832014228942**	**1.25407472935249**	2.0657E-11
9	**0.15832014228757**	**1.25407472933630**	4.6671E-12
10	**0.15832014228746**	**1.25407472933664**	4.8262E-12
11	**0.15832014228746**	**1.25407472933662**	4.8155E-12
12	**0.15832014228746**	**1.25407472933662**	4.8155E-12

Maurer and Osmolovskii [20] have employed the code BNDSCO [22] that is based on shooting methods and reported an optimal bang-bang solution with one switching time τ_s =0.1583201376 and final time τ_f =1.25407473 for this problem.

The results τ_s =**0.15832008477693** and τ_f =**1.25407514033795** obtained by MLGI for N =5 are in excellent agreement with their reported results, while our direct method is significantly simpler in comparison to the indirect method used by them.

The results of MLGI method for N =5, is shown in Figure 2. The resulted control values are $u^1 = 1$ and $u^2 = -1$ in first and second subdomains, respectively.

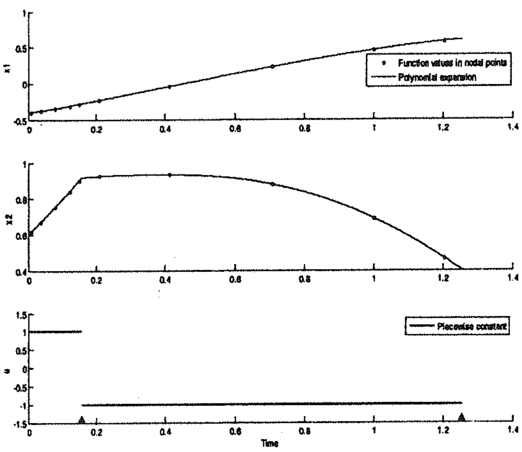

Figure 2: numerical results of MLGI method for Van der Pol problem, $N = 5$.

The results for N =10 provide the final time τ_f =1.2540747293366 with 14 digit precision, since by increasing N, we can see no variation in these digits.
To verify the results, the differential equations of the problem were solved by MATLAB function ode45 using the control, switching times and final time that had been provided by MLGI method. Then, the final states obtained by ode45 $\mathbf{x}_{f\,(ode45)}$, were compared with given final constraints $\mathbf{x}_f$ of the problem by computing the error norm $\|E\| = \|\mathbf{x}_f - \mathbf{x}_{f\,(ode45)}\|$.
For N =5, The verification showed that the final constraints are satisfied with accuracy of 6.8604E-07.

B. Time-optimal control for F-8 aircraft

To show power and accuracy of our method, MLGI was employed for stabilization of a modern high performance aircraft with bang-bang control.
The longitudinal nonlinear equations of the motion for the F-8 aircraft with nonlinearity in states and control are given as, [23].

$$\begin{aligned}\dot{x}_1 = &-0.877x_1 + 0.47x_1^2 + 3.846x_1^3 - \\ &0.019x_2^2 + x_3 - 0.088x_1x_3 - x_1^2x_3 \\ &-0.215u + 0.28x_1^2u + 0.47x\,u^2 \\ &+0.63u^3\end{aligned} \quad (55)$$

$$\dot{x}_2 = x_3 \quad (56)$$

$$\begin{aligned}\dot{x}_3 = &-4.208x_1 - 0.47x_1^2 - 0.396x_3 - 3.564x_1^3 \\ &-20.967u + 6.265x_1^2u + 46x_1u^2 + 61.4u^3\end{aligned} \quad (57)$$

where x_1 is the angle of attack (in radians), x_2 is the pitch angle (in radians), x_3 is the pitch rate, and the control input u is the elevator deflection angle (in radians). This model, originally derived by Grarrd and Jordan [23], has been used in various control studies [9],[7],[8] have considered the stabilization of the aircraft with bang-bang control.
The boundary conditions are considered as $\mathbf{x}_0 = [26.7^\circ, 0, 0]$ and $\mathbf{x}_f = [0,0,0]$, respectively. The control input is bounded by lower and upper bounds, $u_{\min} = -3^\circ$ and $u_{\max} = 3^\circ$. An approximate value of 0.05236 radians for 3° and an exact value of $26.7 * \pi / 180 rad$ for 26.7°, was used. Here, the time-optimal control of this system was considered.
We applied the LGI method with N =30 to this problem. The resulted control function is shown in Figure 3. In this figure, the Gibbs phenomenon is observed. The knowledge of this structure provided by this method is worthy.

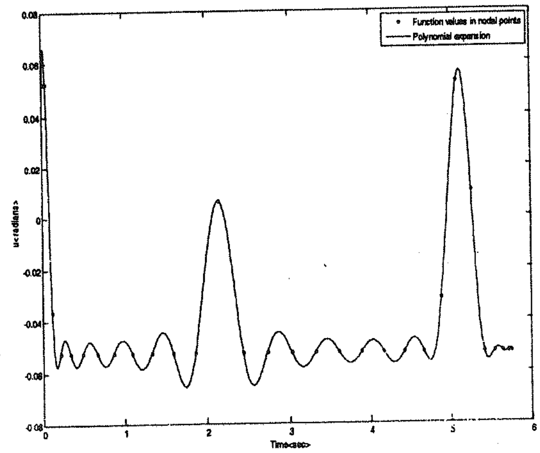

Figure 3: Control history of LGI method for F-8 aircraft, N =30, the dot symbols are the control function values in the LG nodes resulted by solving NLP problem. The solid line is the corresponding Lagrange expansion ($u^N(t)$).

To find the optimal switching times, final time and controls, we employed MLGI method by using the above LGI results as initial guess. Considering N=10, we obtained a solution with 5 switching. The resulted final time is τ_f =5.74216978701382 and the switching times and controls are as follows: τ_s^1=0.10292206601323, τ_s^2=2.03084964621018, τ_s^3=2.19771496239478, τ_s^4=4.94108126538256, τ_s^5=5.27100440978152, u^1=0.05236, u^2=-0.05236, u^3=0.05236, u^4=-0.05236, u^5=-0.05236, u^6=-0.05236. The results of MLGI method for N =10 with five switching, is shown in Figure 4.

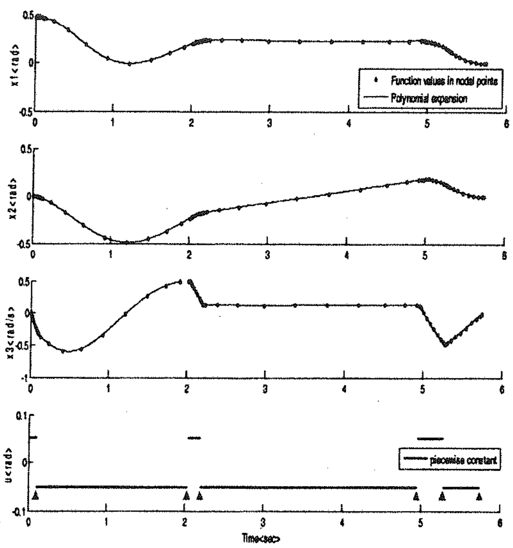

Figure 4: numerical results of MLGI method for F-8 aircraft, five switching, N =10.

As other local minimizer algorithms, different local solutions may be obtained with different initial guesses. Considering initial guesses τ_s^1=1, τ_s^2=2, τ_s^3=3 (three switching), τ_f=4 and N=15, a superior result of τ_f=3.78151586849429 was obtained with the following switching times and controls: τ_s^1=1.13276380315143, τ_s^2=1.48025005808784, τ_s^3=3.08914171082237, u^1=0.05236, u^2=-0.05236, u^3=0.05236, u^4=-0.05236. The above final and switch times are in excellent agreement with the results reported by Kaya et al. using STC method [24]. Until now, less than this final time has not been reported. The results of MLGI method for N=15 with three switching, are shown in Figure 5. The results of MLGI method with three switching for various values of N are reported in Table 2.

TABLE 2:
MLGI METHOD RESULTS FOR F-8 AIRCRAFT FOR VARIOUS VALUES OF N

N	τ_f	$\|E\|$
5	3.82809229097944	5.9827e-01
10	3.78138514314386	2.5366E-04
15	3.78151586849429	1.0217E-06
20	3.78151658861839	6.4136E-09
25	3.78151659255408	4.7032E-11
30	3.78151659258167	1.3006E-11
35	3.78151659258095	1.1991E-12

In addition, we verified the solutions using ode45. The results are presented in Table 2. Here, since the final constraint is $\mathbf{x}_f = [0,0,0]$, the norm of error E is equal to $\|\mathbf{x}_{f\,(ode\,45)}\|$ which is the distance of the dynamic variables of the aircraft to the origin. The results show that N =15 is enough to obtain the optimal control which satisfies the final constraint with good accuracy of 1E-06.

Different researchers have solved this problem, and reported different solutions [9],[7],[8],[24]. Kaya and Noakes [7] have reported a final time τ_f=6.6967 and τ_f=6.3867 with two and three switching, respectively. Lee et al. [9] have reported a solution with four switching and τ_f=6.035256.

A solution with five switching and τ_f=5.74217 has been reported by Kaya and Noakes [8]. Finally, Kaya et al. [24] have reported a superior new local minima with three switching and τ_f=3.781517 that is significantly smaller than the other reported results. However, it is possible that there exist further local solutions. The results of the MLGI method are in excellent agreement with the best reported results until now.

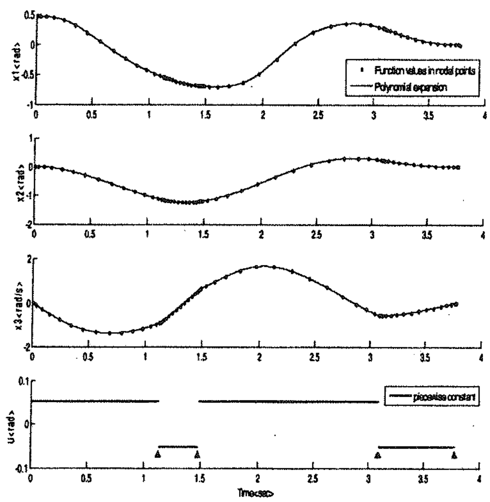

Figure 5: numerical results of MLGI method for F-8 aircraft, three switching, N =15

6. DISCUSSION AND CONCLUSION

In this paper, an accurate and efficient direct method, called MLGI method, is proposed to solve the time-optimal switching control problem. This method employs the spectral collocation technique to convert the control problem to a nonlinear programming problem, which then can be solved by well-developed NLP solvers. To illustrate the method, it was applied to the Van der Pol and F-8 aircraft time-optimal control problems. The results are in excellent agreement with the best reported solutions in the literature for these problems. The method provides highly accurate results with small number of nodes and spectral accuracy. This increases the time efficiency of the algorithm. In addition, for a desired accuracy, the size of the resulted NLP problem is small. Therefore, it can be solved by simple nonlinear programming solvers. The method is computationally attractive.

In comparison to indirect method that are used to solve time-optimal control problem, the proposed method benefits from good convergence property and robustness. The MLGI can be used without any changes in the code for solving different problems. Only a MATLAB function that defines the state equations of the problem should be changed. This is a beneficial point in comparison to indirect methods and Lagrangian-based methods where the optimality necessary conditions should be derived analytically for each new problem.

In general, solutions provided by MLGI method are piecewise-constant control. Bang-bang control solutions that are obtained by this method for studied numerical examples are special cases of piecewise-constant control.

Although in this paper we have applied the MLGI method to solve time-optimal control problems, we should emphasis that the proposed method can be used for general Bolza cost functions. This can be achieved by discretization of the Bolza integral term using Legendre-Guess quadrature

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