A Locally Symmetric AlmostK ähler Einstein Structure on the Cotangent Bundle of a Riemannian Manifold

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ABSTRACT

We introduce a Riemannin metric of diagonal type on the cotangent bundle of a Riemannian manifold and show that T^*M with this metric is locally symmetric Einstein manifold. Also, we obtain a locally symmetric Kähler Einstein structure on the cotangent bundle of a Riemannian manifold of negative constant sectional curvature. Similar results are obtained on a tube around zero section in the cotangent bundle, in the case of a Riemannian manifold of positive constant sectional curvature.

KEYWORDS

Almost complex structure, cotangent bundle, Kähler Einstein metric, locally symmetric Riemannian manifold.

1. Introduction

The differential geometry of the cotangent bundle T^*M of a Riemannian manifold (M,g) is almost similar to that of the tangent bundle TM. However, there are some differences. This is because the lifts (vertical, complete, horizontal, etc.) to T^*M cannot be defined just like in the case of TM.

In [6] V. Oproiu and D.D. porosniuc have obtained a natural Kähler Einstein structure (G,J) of diagonal type induced on T^*M from the Riemannian metric g. The obtained Kähler structure on T^*M, depends on parameters u, which is a smooth function depending on the energy density τ on T^*M.

In this paper, we obtain a Kähler structure on T^*M, which depends on parameters α, β, u, where $\alpha, \beta \in \mathbb{R}$. The vertical distribution VT^*M and the horizontal distribution HT^*M are orthogonal to each other but the dot products induced on them from G are not isomorphic (isometric).

After that, we obtain that G is Hermitian with respect to J and it follows that the fundamental 2-form Ω, associated to the almost Hermitian structure (G,J) is the fundamental form defining the usual symplectic structure on T^*M, hence it is closed.

From the integrability condition for J it follows that the base manifold M must have constant sectional curvature c and the parameter u must be constant.

If the constant sectional c is negative then we obtain a locally symmetric Kähler Einstein structure defined on the whole T^*M. If the constant sectional curvature c of M is positive then we get a similar structure defined on a tube around zero section in T^*M.

The manifolds, tensor fields and geometric objects we consider in this paper, are assumed to be differentiable of class C^∞ (i.e., smooth). We use the computations in local coordinates but many results from this paper may be expressed in an invariant form. The well-known summation convention is used throughout this paper, the range for the indices h,i,j,k,l,r,s being always $\{1,\dots,n\}$ (see [5], [7]). The module of smooth vector fields on T^*M shall be denoted by $\Gamma(T^*M)$.

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1. Preliminar

Let (T^*M,π,M) be the cotangent bundle, where M is an n-dimensional smooth manifold. If (U,φ) is a local chart on M and (x^i) are the coordinates of a point $p\in M$, $p\in\varphi^{-1}(x)\in U$, then a point $u\in\pi^{-1}(U)$, $\pi(u)=p$ has the coordinates (x^i,p_i), $(i=1,\dots,n)$. The natural basis of the module $\Gamma(T^*M)$ is given by $(\partial_i=\frac{\partial}{\partial x^i},\partial^r=\frac{\partial}{\partial p_r})$.

An M-tensor field of type (r,s) on T^*M is defined by set of n^{r+s} components (functions depending on x^i and p_i), with r upper indices and s lower indices, assigned to induced local charts $(\pi^{-1}(U),\Phi)$ on T^*M, such that the local coordinate change rule is that of the local coordinate components of a tensor field of type (r,s) on the base manifold M. A usual tensor field of type (r,s) on M may be thought of as an M-tensor field of type (r,s) on T^*M. If the considered tensor field on M is covariant only, the corresponding M-tensor field on T^*M may be identified with the induced (pullback by π) tensor field on T^*M.

Some useful M-tensor fields on T^*M may be obtained as follows. Let $u,v:[0,\infty]\to\mathbb{R}$ be a smooth functions and let $\|p\|^2=g^{-1}_{\pi(p)}(p,p)$ be the square of the norm of the cotangent vector $p\in\pi^{-1}(U)$. The components $g_{ij}(\pi(p))$, p_i, $u(\|p\|^2)p_ip_j$ define M-tensor fields of type $(0,2)$, $(0,1)$, $(0,2)$ on T^*M, respectively. Similarly, the components $g^{kl}(\pi(p))$, $p^i=p_hg^{hi}$, $v(\|p\|^2)p^kp^l$ define M-tensor fields of type $(2,0)$, $(1,0)$, $(2,0)$ on T^*M, respectively. Of course, all the components considered above are in the induced local chart $(\pi^{-1}(U),\Phi)$.

The Levi Civita connection ∇ of g defines a direct sum decomposition

$$TT^*M=VT^*M\oplus HT^*M, \tag{1}$$

of the tangent bundle to T^*M into vertical distribution $VT^*M=\ker\pi_*$ and the horizontal distribution HT^*M.

If $(\pi^{-1}(U),\Phi)=(\pi^{-1}(U),x^1,\dots,x^n,p_1,\dots,p_n)$ is a local chart on T^*M, induced from the local chart $(U,\varphi)=(U,x^1,\dots,x^n)$, the local vector fields $\partial^1=\frac{\partial}{\partial p_1},\dots,\partial^n=\frac{\partial}{\partial p_n}$ on $\pi^{-1}(U)$ defines a local frame for VT^*M over $\pi^{-1}(U)$ and the local vector fields $\delta_1=\frac{\delta}{\delta x^1},\dots,\delta_n=\frac{\delta}{\delta x^n}$ define a local frame for HT^*M over $\pi^{-1}(U)$, where

$$\delta_i=\partial_i+p_h\Gamma^h_{ik}\partial^k,$$

and $\Gamma^h_{ik}(\pi(p))$ are the Christoffel symbols of g.

The set of vector fields $(\partial^1,\dots,\partial^n,\delta_1,\dots,\delta_n)$ defines a local frame on T^*M, adapted to the direct sum decomposition (1).

We consider

$$\tau=\frac{1}{2}\|p\|^2=\frac{1}{2}g^{-1}_{\pi(p)}(p,p)=\frac{1}{2}g^{ik}(x)p_ip_k, \tag{2}$$

$$p\in\pi^{-1}(U)$$

the energy density defined by g in the cotangent vector p. We have for all $p\in T^*M$.

From now on, we shall work in a fixed local chart (U,φ) on M and in the induced local chart $(\pi^{-1}(U),\Phi)$ on T^*M.

3. An almost Kähler structure on T^*M

Consider a real valued smooth function u defined on $[0,\infty)\subset\mathbb{R}$ and real constants α and β. We define the following symmetric M- tensor field of type (0,2) on T^*M having the components

$$G_{ij}(p)=\frac{1}{\beta}g_{ij}(\pi(p))+\frac{u(\tau)}{\alpha\beta}p_ip_j. \tag{3}$$

It follows easily that the matrix (G_{ij}) is positive definite if and only if $\alpha,\beta>0$, $\alpha+2\tau u>0$. The inverse of this matrix has the entries

$$H^{kl}(p)=\beta g^{kl}(\pi(p))+v(\tau)p^kp^l, \tag{4}$$

where

$$v=-\frac{u\beta}{\alpha+2\tau u}. \tag{5}$$

The components H^{kl} define symmetric M- tensor field of type (2,0) on T^*M. It follows:

Remark. If the matrix (G_{ij}) is positive definite, then matrix (H^{kl}) is positive definite, too.

Using the M - tensor fields defined by G_{ij} and H^{kl}, the following Riemannian metric may be considered on T^*M:

$$G = G_{ij}dx^i dx^j + H^{ij}\delta p_i \delta p_j, \quad (6)$$

where $\delta p_i = dp_i - p_h \Gamma^h_{ik} dx^k$ is the the covariant differential of p_i with respect to the Levi Civita connection ∇ of g. Equivalently, we have

$$G(\delta_i, \delta_j) = G_{ij}, \quad G(\partial^i, \partial^j) = H^{ij},$$
$$G(\delta_i, \partial^j) = G(\partial^i, \delta_j) = 0.$$

Note that HT^*M and VT^*M are orthogonal to each other with respect to G, but the Riemannian metrics induced from G on HT^*M, VT^*M are not the same, so the considered metric G on T^*M is not a metric of Sasaki type. Note also that the system of 1-forms $(dx^1,\dots,dx^n,\delta p_1,\dots,\delta p_n)$ defines a local frame on T^*T^*M, dual to the local frame $(\delta_1,\dots,\delta_n,\partial^1,\dots,\partial^n)$ adapted to the direct sum decomposition (1).

Next, an almost complex structure J is defined on T^*M by the same M- tensor fields G_{ij}, H^{kl}, expressed in the adapted local frame by

$$J(\delta_i) = G_{ik}\partial^k, \quad J(\partial^i) = -H^{ik}\delta_k \quad (7)$$

From the property of the M - tensor field H^{kl} to be defined by the inverse of the matrix defined by the components of the M - tensor field G_{ij}, it follows easily that J is an almost complex structure on T^*M.

Theorem 1. (T^*M, G, J) is an almost Kähler manifold.

Proof: Since the matrix (H^{kl}) is the inverse of the matrix (G_{ij}), it follows easily that

$$G(J\delta_i, J\delta_j) = G(\delta_i, \delta_j),\ G(J\partial^i, J\partial^j) = G(\partial^i, \partial^j),$$
$$G(J\delta_i, J\partial^j) = G(\delta_i, \partial^j) = 0.$$

Hence

$$G(JX,JY)=G(X,Y), \quad \forall X,Y \in \Gamma(T^*M).$$

Thus (T^*M, G, J) is an almost Hermitian manifold.

The fundamental 2-form associated with this almost Hermitian structure is Ω, defined by

$$\Omega(X,Y) = G(X,JY), \quad \forall X,Y \in \Gamma(T^*M).$$

By a straightforward computation we get

$$\Omega(\delta_i,\delta_j) = 0,\ \Omega(\partial^i,\partial^j) = 0,\ \Omega(\partial^i,\delta_j) = \delta^i_j.$$

Hence

$$\Omega = \delta p_i \wedge dx^i = dp_i \wedge dx^i, \quad (8)$$

due to the symmetry of $p_h \Gamma^h_{ij}$. It follows that Ω does coincide with the fundamental 2-form defining the usual symplectic structure on T^*M. Of course, we have $d\Omega = 0$, i.e., Ω is closed. Therefore, (T^*M, G, J) is an almost Kähler manifold. □

4.A Kähler structure on T^*M

We shall study the integrability of the almost complex structure defined by J on T^*M. To do this, we need the following well-known formulas for the brackets of the vector fields ∂^i, δ_i, $i = 1,\dots,n$

$$[\partial^i, \partial^j] = 0,\ [\partial^i, \delta_j] = \Gamma^i_{jk}\partial^k,\ [\delta_i, \delta_j] = p_h R^h_{kij}\partial^k, \quad (9)$$

where $R^h_{kij}(\pi(p))$ are the local coordinate components of the curvature tensor field of ∇ on M. Of course, the components R^h_{kij} define M-tensor fields of type $(1,3)$ on T^*M.

Lemma 1. The Nijenhuis tensor field of the almost complex structure J on T^*M is given by

$$\begin{cases} N(\delta_i,\delta_j) = -\{\frac{u}{\alpha\beta^2}(\delta^h_i g_{jk} - \delta^h_j g_{ik}) + \\ R^h_{kij}\}p_h\partial^k, \\ N(\delta_i,\partial^j) = -H^{kl}H^{jr}\{\frac{u}{\alpha\beta^2}(\delta^h_i g_{rl} - \delta^h_r g_{il}) + \\ R^h_{lir}\}p_h\delta_k, \\ N(\partial^i,\partial^j) = -H^{ir}H^{jl}\{\frac{u}{\alpha\beta^2}(\delta^h_l g_{rk} - \delta^h_r g_{lk}) + \\ R^h_{klr}\}p_h\partial^k. \end{cases} \quad (10)$$

Proof. Recall that the Nijenhuis tensor field N defined by J is given

$$N(X,Y) = [JX,JY] - J[JX,Y] - J[X,JY]$$
$$-[X,Y], \quad \forall X,Y \in \Gamma(T^*M).$$

Then, we have $\delta_k \tau = 0$, $\partial^k \tau = p^k$ and $\nabla_i G_{jk} = 0$, $\nabla_i H^{jk} = 0$, where

$$\begin{cases} \nabla_i G_{jk} = \delta_i G_{jk} - \Gamma^l_{ij}G_{lk} - \Gamma^l_{ik}G_{lj}, \\ \nabla_i H^{jk} = \delta_i H^{jk} + \Gamma^j_{il}H^{lk} + \Gamma^k_{il}H^{lj}. \end{cases}$$

The above expressions for the components of N can be obtained by a quite long, straightforward computation. □

Theorem 2. Let M be a connected Riemannian manifold and $\dim M \geq 3$, then the almost complex structure J on T^*M is integrable if and only if the manifold M has constant sectional curvature c and the function u is given by

$$u = -c\alpha\beta^2. \quad (11)$$

Proof. From the condition $N = 0$, one obtains

$$\{\frac{u}{\alpha\beta^2}(\delta_i^h g_{jk} - \delta_j^h g_{ik}) + R_{kij}^h\} p_h = 0.$$

Differentiating with respect to p_r, taking $p_h = 0 \ \forall h \in \{1,\dots,n\}$, it follows that the curvature tensor field of ∇ has the expression

$$R_{kij}^r = -\frac{u(0)}{\alpha\beta^2}(\delta_i^r g_{jk} - \delta_j^r g_{ik}).$$

Using the Schur theorem it follows that (M,g) has the constant sectional curvature $c = -\frac{u(0)}{\alpha\beta^2}$. Then, we obtain the expression (11) of u.

Conversely, if (M,g) has constant curvature c and u is given by (11), it follows in a straightforward way that $N = 0$. □

It should be noted that the function u must fulfill the condition

$$\alpha + 2\tau u = \alpha(1 - 2c\beta^2\tau) > 0, \quad \alpha,\beta > 0. \quad (12)$$

Therefore, the theorem can be stated as follows:

Theorem 3. *Let M be a Riemannian manifold with constant sectional curvature c and the function $u = -c\alpha\beta^2$, then*

i) If $c < 0$ then (T^*M, G, J) is a Kähler manifold.

ii) If $c > 0$ then (T_α^*M, G, J) is a Kähler manifold, where T_α^*M is the tube around zero section in T^*M defined by the condition $0 \leq \| p \|^2 < \frac{1}{c\beta^2}$.

iii) If $c = 0$ then (M,g) is a flat manifold and G becomes a flat metric on T^*M.

The components of the Kähler metric G on T^*M are

$$\begin{cases} G_{ij} = \frac{1}{\beta} g_{ij} - c\beta p_i p_j, \\ H^{ij} = \beta g^{ij} + \frac{c\beta^3}{1 - 2c\beta^2\tau} p^i p^j. \end{cases} \quad (13)$$

5. A Kähler Einstein structure on T^*M

In this section, we shall study the property of the Kähler manifold (T^*M, G, J) to be Einstein.

The Levi Civita connection $\overline{\nabla}$ of the Riemannian manifold (T^*M, G) is determined by the conditions

$$\overline{\nabla} G = 0, \quad T = 0,$$

where T is its torsion tensor field. The explicit expression of this connection is obtained from the formula

$$\begin{aligned} 2G(\overline{\nabla}_X Y, Z) &= X(G(Y,Z)) + Y(G(X,Z)) \\ &- Z(G(X,Y)) + G([X,Y],Z) - G([X,Z],Y) \\ &- G([Y,Z],X); \\ &\forall X,Y,Z \in \Gamma(T^*M). \end{aligned}$$

The final result can be stated as follows:

Lemma 2. The Levi Civita connection $\overline{\nabla}$ of G has the following expression in the local adapted frame $(\delta_1,\dots,\delta_n,\partial^1,\dots,\partial^n)$:

$$\begin{cases} \overline{\nabla}_{\partial^i} \partial^j = Q_h^{ij} \partial^h, \ \overline{\nabla}_{\delta_i} \partial^j = -\Gamma_{ih}^j \partial^h + P_i^{hj} \delta_h, \\ \overline{\nabla}_{\partial^i} \delta_j = P_j^{hi} \delta_h, \ \overline{\nabla}_{\delta_i} \delta_j = -\Gamma_{ij}^h \delta_h + S_{hij} \partial^h, \end{cases} \quad (14)$$

where Q_h^{ij}, P_j^{hi}, S_{hij} are M-tensor fields on T^*M, defined by

$$\begin{cases} Q_h^{ij} = \frac{1}{2} G_{hk} (\partial^i H^{jk} + \partial^j H^{ik} - \partial^k H^{ij}), \\ P_j^{hi} = \frac{1}{2} H^{hk} (\partial^i G_{jk} - H^{il} R_{ljk}^r p_r), \\ S_{hij} = -\frac{1}{2} G_{hk} \partial^k G_{ij} + \frac{1}{2} R_{hij}^r p_r). \end{cases} \quad (15)$$

After replacing the expressions of the involved M-tensor fields (equation (12)), we obtain

$$Q_h^{ij} = c\beta H^{ij} p_h, \ P_j^{hi} = -c\beta H^{hi} p_j, \ S_{hij} = c\beta G_{hj} p_i.$$
(16)

The curvature tensor field K of the connection $\overline{\nabla}$ is obtained from the well-known formula

$$K(X,Y)Z = \overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]} Z,$$
$$\forall X,Y,Z \in \Gamma(T^*M).$$

The components of curvature tensor field K with respect to the adapted local frame $(\delta_1,\dots,\delta_n,\partial^1,\dots,\partial^n)$ are obtained easily:

$$\begin{cases} K(\delta_i,\delta_j)\delta_k = c\beta(\delta_i^h G_{jk} - \delta_j^h G_{ik})\delta_h, \\ K(\delta_i,\delta_j)\partial^k = c\beta(\delta_j^k G_{hi} - \delta_i^k G_{hj})\partial^h, \\ K(\partial^i,\partial^j)\delta_k = c\beta(\delta_k^j H^{hi} - \delta_k^i H^{hj})\delta_h, \\ K(\partial^i,\partial^j)\partial^k = c\beta(\delta_h^i H^{jk} - \delta_h^j H^{ik})\partial^h, \\ K(\partial^i,\delta_j)\delta_k = c\beta\delta_j^i G_{hk}\partial^h, \\ K(\partial^i,\delta_j)\partial^k = -c\beta\delta_j^i H^{hk}\delta_h. \end{cases} \quad (17)$$

The Ricci tensor field Ric of $\overline{\overline{\nabla}}$ is defined by the formula:

$$Ric(Y,Z) = trace(X \to K(X,Y)Z), \quad \forall X,Y,Z \in \Gamma(T^*M).$$

It follows

$$\begin{cases} Ric(\delta_i,\delta_j) = cn\beta G_{ij}, \\ Ric(\partial^i,\partial^j) = cn\beta H^{ij}, \\ Ric(\partial^i,\delta_j) = Ric(\delta_j,\partial^i) = 0. \end{cases}$$

Thus

$$Ric = cn\beta G. \quad (18)$$

By using expression (17), we have computed the covariant derivatives of curvature tensor field K in the local adapted frame (δ_i,∂^i) with respect to the connection $\overline{\overline{\nabla}}$ and we obtained in all twelve cases the result is zero, i.e., $\overline{\overline{\nabla}}K = 0$.

Hence, we may state our main result.

Theorem 5. *Assume that the Riemannian manifold (M,g) has constant sectional curvature c, condition (14) are fulfilled and the components of the metric G are given by (20), then*

1. If $c<0$ then (T^*M,G,J) is a locally symmetric Kähler Einstein manifold.

2. If $c>0$ then (T_a^*M,G,J) is a locally symmetric Kähler Einstein manifold.

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