Stability Analysis of Self-Feedback Neural Network Structures

Mahmood Amiri[i]; Mohammad Bagher Menhaj[ii]; Ali Fallah[iii]

ABSTRACT

This paper investigates the stability analysis of self-feedback neural network structures, i.e., some kind of recurrent neural networks comprised of self-feedback neurons. Theoretically, it will be shown how the number of fixed points and their stability properties are influenced by the network parameter values. A number of explicit equations among network parameters such as self-feedback coefficients, input weights and the number of equilibrium points are obtained and it will be proven that each neuron has, at most, three fixed points, such that two of which are asymptotically stable and the other is unstable. Since the dynamical equations that describe these types of networks are uncoupled, it is easy to generalize these results to higher order dimensions. Several simulations are provided to demonstrate the effectiveness of the analytical results presented in this paper.

KEYWORDS

Self-feedback neural network structure, equilibrium points, asymptotic stability.

1. INTRODUCTION

In recent years, neural networks have been widely used because of their remarkable capabilities as generalization; parallel processing, nonlinear system modeling, adaptation, and function approximating [1-3]. In the conventional structure of an artificial neural network, a neuron receives its input either from other neurons or from external inputs (input vector). A weighted sum of these inputs constitutes the argument of a fixed nonlinear activation function. The resulting value of the activation function is the neural output. This class of neural networks, namely FFNN (Fig.1.a), is a static mapping and experiences some difficulties in representing a nonlinear dynamical system [4-6]. On the other hand, recurrent neural networks (RNN) are capable of approximating dynamically and are more appropriate than FFNN when applied to a nonlinear dynamical system [7-8]. There are many RNN models that consist of both feed-forward and feedback connections between layers and neurons forming complicated dynamics [9].

Diagonal recurrent neural network (DRNN), a modified form of fully recurrent neural network (FRNN, Fig.1.b), was firstly put forward by Ku and Lee [10]. DRNN is a two-layer network (Fig.1.c), where the hidden layer contains self-feedback neurons while the output layer is comprised of linear neuron. The self-feedback connection of hidden neurons ensures that the output of DRNN contains the whole past information of the system even if the inputs of the DRNN are only the present states and inputs of the system. Since there is no interlink between neurons in the hidden layer, the DRNN has considerably fewer weights than the FRNN and the network is noticeably simplified [11].

Hopfield presented continuous-time feedback neural networks, which provided a way of storing analog patterns [12]. Storage of analog pattern vectors with real-valued components using feedback neural networks is of great interest, since in applications such as associative memories, pattern recognition, vector quantization and image processing, the patterns are originally in analog form and costly quantization may be avoided [13-15]. The authors of this paper in [16] investigated the stability analysis of auto-associative memory and demonstrated how the stability properties of fixed points depend on network parameters values.

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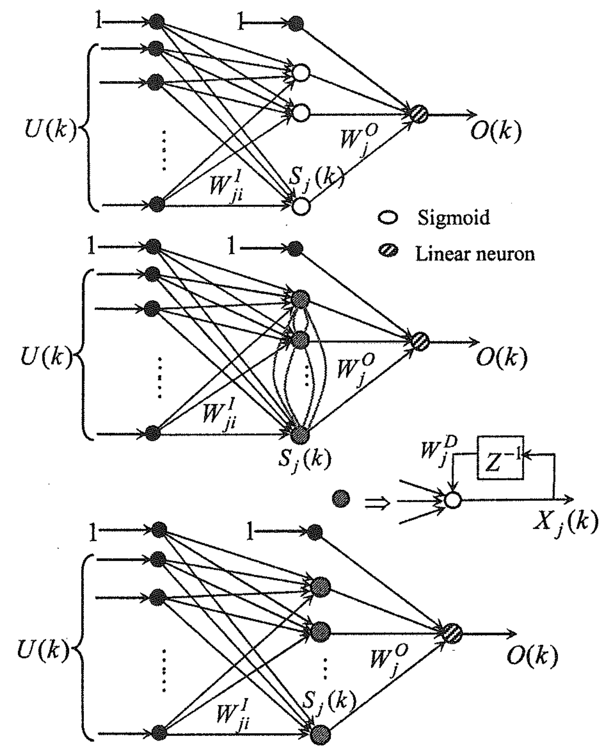


Figure 1: Three structures for neural networks: a) Feed forward neural network b) Fully recurrent neural network c) Diagonal recurrent neural network.

The rest of this paper is organized as follows: In Section 2, some relevant definitions and theorems will be introduced and then the DRNN model will be described. The stability of a self-feedback neuron and its generalization to higher dimension will be fully investigated in Section 3. In Section 4, the experimental results will be presented. Finally, Section 5 concludes the paper.

2. Method

A. Preliminaries

In general, dynamical systems may be discrete or continuous, depending on whether they are described by differential or difference equations. The difference equation for a general time-invariant discrete dynamical system can be written as:

$$x_{k+1} = f(x_k) \quad k = 0,1,... \qquad (1)$$

where $f : \Re \rightarrow \Re$ can be a linear or nonlinear function of x_k. The following definitions and theorems are of interest in (1) [7]:

Definition 1. A point $\overline{x} \in \Re$ is an *equilibrium point* for the dynamical system (1), or a *fixed point* for map f, if $f(\overline{x}) = \overline{x}$.

Definition 2.a. A fixed point $\overline{x}$ of (1) is said to be *stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $|x_0 - \overline{x}| < \delta$, the point $\overline{x}$ satisfy $|x_k - \overline{x}| < \varepsilon$.

Definition 2.b. A fixed point $\overline{x}$ of (1) is said to be *unstable* if it is not stable.

Definition 2.c. A fixed point $\overline{x}$ of (1) is said to be *asymptotically stable* or an *attracting* fixed point of the function f if it is stable and, in addition, there exists $r > 0$ such that for all x_0 satisfying $|x_0 - \overline{x}| < r$, then sequence x_k satisfy $\lim_{k\to\infty} x_k = \overline{x}$.

Theorem 1. Let $f : \Re \rightarrow \Re$ be continuously differentiable in a neighborhood of $\overline{x}$, then $\overline{x}$ is asymptotically stable if $|f'(\overline{x})| < 1$ and is unstable if $|f'(\overline{x})| > 1$.

Theorem 2. Assume that f has a continuous second derivative at an equilibrium point $\overline{x}$ and suppose that $f'(\overline{x}) = 1$ and $f''(\overline{x}) \neq 0$. Then $\overline{x}$ is a semi-stable equilibrium solution of (4). In particular, the following statements are true.

a) If $f''(\overline{x}) < 0$, then $\overline{x}$ is a semi-stable from above equilibrium solution of (1).

b) If $f''(\overline{x}) > 0$, then $\overline{x}$ is a semi-stable from below equilibrium solution of (1).

B. The Diagonal Recurrent Neural Network

The architecture of the DRNN model is depicted in Fig.1. The mathematical description is as follows [8]:

$$S_j(k) = W_j^D X_j(k-1) + \sum_{i=1}^{n} W_{ij}^I u_i(k) \qquad (2)$$

$$X_j(k) = f(S_j(k)) \qquad (3)$$

$$O(k) = \sum_{j=1}^{m} W_j^O X_j(k) \qquad (4)$$

where $u_i(k)$ $(i = 1,...,n)$ denote the external input, and $S_j(k)$, $X_j(k)$ $(j = 1,...,m)$ are the input and output of the j^{th} neuron of the output layer, respectively. $f(\lambda)$ is the activation function defined by $f(\lambda) = 1/(1+e^{-\lambda})$. W_{ij}^I, W_j^D are connection weights from input to output layer and within the output layer, respectively.

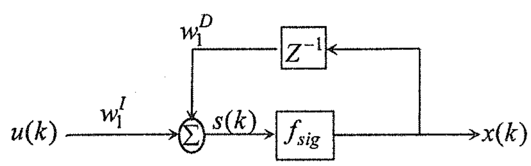


Figure 2: Single self-feedback neuron.

3. Stability Analysis

A. Single Self-Feedback Neuron

The self-feedback neuron model is illustrated in Fig. 2. It can be described by the following equations:

$$s(k) = w_1^I u(k) + w_1^D f(s(k-1)) \quad (5)$$

$$x(k) = f(s(k)) \quad (6)$$

From (5) using definition (1), $s(k)$ at the fixed point can be represented as:

$$\bar{s} = w_1^I u(k) + w_1^D f(\bar{s}) \quad (7)$$

which yields:

$$f(\bar{s}) = \left(\frac{1}{w_1^D}\right)\bar{s} - \left(\frac{w_1^I}{w_1^D}\right)u(k) \quad (8)$$

Now let:

$$a = 1/w_1^D \quad (9)$$

and

$$b = \left(w_1^I / w_1^D\right)u(k) = w_1^I.a.u(k) \quad (10)$$

Therefore:

$$f(\bar{s}) = a\bar{s} - b \quad (11)$$

Suppose that $\bar{x}$ is the neuron output at $\bar{s}$:

$$\bar{x} = f(\bar{s}) \quad (12)$$

and let $a.\bar{s} - b = g(\bar{s})$.

There are three different situations for fixed points, based on different intersection situations between line *g(s)* and curve *f(s)*: 1) Two fixed points; 2) Three fixed points; and 3) One fixed point.

A.1 Two fixed points

In this case (see Fig. 5), the slopes of the two functions $f(\bar{s})$, $g(\bar{s})$ are the same. Thus:

$$f'(\bar{s}) = a \Rightarrow f(\bar{s})[1 - f(\bar{s})] = a \quad (13)$$

Since $\bar{x} = f(\bar{s})$, we can rewrite (13) as:

$$\bar{x}^2 - \bar{x} + a = 0 \quad (14)$$

which results:

$$\bar{x}_1 = \frac{1+\sqrt{1-4a}}{2} \quad , \quad \bar{x}_2 = \frac{1-\sqrt{1-4a}}{2} \quad (15)$$

Therefore, we should have $0 < a < 0.25$ or $w_1^D > 4$. To determine the value of *b*, we use the following:

$$\bar{x} = f(\bar{s}) \quad \Rightarrow \bar{s} = -\ln\left(\frac{1-\bar{x}}{\bar{x}}\right) \quad (16)$$

By substituting $\bar{x}_1$ and $\bar{x}_2$ from (15) into (16), we obtain:

$$\bar{s}_2 = \ln\left(\frac{1-\sqrt{1-4a}}{1+\sqrt{1-4a}}\right) \quad \bar{s}_1 = \ln\left(\frac{1+\sqrt{1-4a}}{1-\sqrt{1-4a}}\right) \quad (17)$$

From (11), (15) and (17) we can conclude:

$$b_1 = a\bar{s}_1 - \bar{x}_1 = a\ln\left(\frac{1+\sqrt{1-4a}}{1-\sqrt{1-4a}}\right) - \frac{1+\sqrt{1-4a}}{2}$$

$$b_2 = a.\bar{s}_2 - \bar{x}_2 = a.\ln\left(\frac{1-\sqrt{1-4a}}{1+\sqrt{1-4a}}\right) - \frac{1-\sqrt{1-4a}}{2} \quad (18)$$

For stability analysis of each of these fixed points, we combine (5) and (6) and rewrite the result in terms of *a* and *b* as follows:

$$x(k) = f\left(\frac{1}{a}x(k-1) + \frac{b}{a}\right) \quad (19)$$

Denote $p(x) = f(\frac{1}{a}x + \frac{b}{a})$ to obtain:

$$p'(\bar{x}) = \frac{1}{a}\bar{x}(1-\bar{x}) \quad (20)$$

$$p''(\bar{x}) = \frac{1}{a^2}\bar{x}(1-\bar{x})(1-2\bar{x}) \quad (21)$$

If we set (20) equal to one, we will obtain (14). So, in this case, we should use Theorem (2) to determine the stability of fixed points.

If $p''(\bar{x}) > 0$, then according to (21) the fixed point, which is smaller than 0.5, is semi-stable from below, and the other fixed point is asymptotically stable. If $p''(\bar{x}) < 0$, then the fixed point, which is greater than 0.5, is semi-stable from above, and the other one is asymptotically stable. Since in applications such as control, semi-stability is equivalent to instability, therefore, in this paper, we regard them as unstable equilibrium points.

A.2 Three fixed points

In this case, the line $g(\bar{s}) = a.\bar{s} - b$ has three intersections with $f(\bar{s})$, two of which are asymptotically stable, and the other is unstable. Indeed, the two lateral fixed points are always stable. To determine the condition for *b* that leads to two stable fixed points, we utilize (20) and theorem (1) as follows:

$$|p'(\bar{x})| < 1 \Rightarrow |\bar{x}(1-\bar{x})| < a \quad (22)$$

which yields:

$$\bar{x} > \frac{1+\sqrt{1-4a}}{2} \quad or \quad \bar{x} < \frac{1-\sqrt{1-4a}}{2} \quad (23)$$

The two inequalities in (23) imply the two aforementioned asymptotically stable fixed points.

Since $\bar{x} = a.\bar{s} - b$, we can rewrite (23) as:

$$a\bar{s}_1 - b > \frac{1+\sqrt{1-4a}}{2} \Rightarrow b < a\bar{s}_1 - \frac{1+\sqrt{1-4a}}{2} \quad (24)$$

And

$$a\bar{s}_2 - b < \frac{1-\sqrt{1-4a}}{2} \Rightarrow b > a\bar{s}_2 - \frac{1-\sqrt{1-4a}}{2} \quad (25)$$

Using (24), (25) and (17), we obtain the following condition for this case:

$$a\ln\left(\frac{1-\sqrt{1-4a}}{1+\sqrt{1-4a}}\right) - \frac{1-\sqrt{1-4a}}{2} < b < a\ln\left(\frac{1+\sqrt{1-4a}}{1-\sqrt{1-4a}}\right) - \frac{1+\sqrt{1-4a}}{2} \quad (26)$$

A.3 One fixed point

In order to have one stable fixed point, according to (18) and(26), *b* should satisfy:

$$b > a.\ln\left(\frac{1+\sqrt{1-4a}}{1-\sqrt{1-4a}}\right) - \frac{1+\sqrt{1-4a}}{2} = b_1 \qquad (27)$$

$$b < a.\ln\left(\frac{1-\sqrt{1-4a}}{1+\sqrt{1-4a}}\right) - \frac{1-\sqrt{1-4a}}{2} = b_2 \qquad (28)$$

The aforementioned situations are summarized in Table 1. As we can see from this table, one neuron has at most three fixed points; two of which are asymptotically stable and the other is unstable.

Table 1
Stability analysis of fixed points in self-feedback neuron

Condition $w_1^D > 4,\ w_1^I = \frac{b\,w_1^D}{u}$	Results
$b = b_1$ or $b = b_2$	Two fixed points; One point is semi-stable and the other one is asymptotically stable.
$b_2 < b < b_1$	Three fixed points; Two lateral points are asymptotically stable and the other point is unstable.
$b > b_1$ or $b < b_2$	One asymptotically stable fixed point.

B. Multi-Input, One Self-Feedback Neuron

Fig. 3 illustrates a self-feedback neuron with multiple inputs. The mathematical description is as follows:

$$s(k) = w_1^D f(s(k-1)) + \sum_{i=1}^{n} w_{i1}^I u_i(k)$$
$$x(k) = f(s(k)) \qquad (29)$$

Let $a = \frac{1}{w_1^D}$ and $b = \frac{1}{w_1^D}\sum_i w_{i1}^I u_i(k)$. According to (29), at the fixed point we have:

$$f(\bar{s}) = a\bar{s} - b \qquad (30)$$

which has the same form as (10), though the definition of b is different. This distinction leads to generally different behaviors in multi-input and single-input neurons. By considering Table 1 and applying some mathematical constraints, e.g., by considering identical values for w_{i1}^I, we can determine the stability of fixed points. In this way, we can consider $\sum_i u_i(k)$ as the new input to the single-input self-feedback neuron analyzed in the previous section.

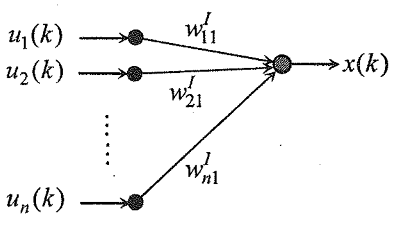


Figure 3: Multi-input, single self-feedback neuron network.

Next, we consider n input nodes with m self-feedback neurons. The situation is illustrated in Fig. 5, which can be described by the following difference equations:

C. Multi-Input, Multi Self-Feedback Neuron

Fig. 4 illustrates a self-feedback neuron with multiple inputs. The mathematical description is as follows:

$$s_j(k) = w_j^D f(s_j(k-1)) + \sum_{i=1}^{n} w_{ij}^I u_i(k)$$
$$x_j(k) = f(s_j(k)) \quad ; \quad j = 1,2\ldots,m \qquad (31)$$

Now letting $a_j = 1/w_j^D$ and $b_j = \frac{1}{w_j^D}\sum_i w_{ij}^I u_i(k) = a_j \sum_i w_{ij}^I u_i(k)$, at the fixed point we will then obtain:

$$f(\bar{s}_j) = a_j \bar{s}_j - b_j \qquad (32)$$

Since equations given in (32) are uncoupled, the system can be considered as a combination of m independent difference equations, each of which has n inputs with one self-feedback neuron, as investigated in the previous section. Because each subsystem has at most three equilibrium points (containing two asymptotically stable equilibria), it is easy to show that the system has at most 3^m equilibria out of which at most 2^m are asymptotically stable.

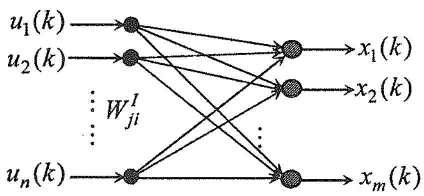


Figure 4: Multi-input, multi-self-feedback neuron network.

Noting that, according to Table 1, we can choose the weight parameters to have just one asymptotically stable equilibrium point, it is better to evaluate the total number of equilibria using the product of the numbers of solutions for each equation (32). By defining

p_i ={*number of solutions of the equation* $f(\bar{s}_j) = a_j\bar{s}_j - b_j$ }

q_i ={*number of asymptotically stable solutions of the equation* $f(\bar{s}_j) = a_j\bar{s}_j - b_j$ }

we have

$$P = \prod_{j=1}^{m} p_j \qquad (33)$$

$$Q = \prod_{j=1}^{m} q_j \qquad (34)$$

where P is the total number of equilibria and Q is the total number of asymptotically stable equilibria of the system.

4. Simulations and Results

In this section, the results of some simulations will be presented to demonstrate the validity of the aforementioned deductions.

A. Single Self-Feedback Neuron

In this experiment, we investigate the stability of fixed points in a self-feedback neuron. Referring to Table 1, w^{1D} can be set to any arbitrary value in $(4,\infty)$, such as *10*. Using (18) we obtain $b_1 = -0.681$ and $b_2 = -0.319$. We assign three values for *b*, corresponding to three situations mentioned in Table 1. Then, we choose an arbitrary value for the input *u*, e.g., *u=1*. In this way, we can calculate the corresponding values for w_1^I using (10), for the above three cases.

Now, we want to adjust w_1^I in order to have two fixed points. As demonstrated in Fig.5, for the case of two fixed points, we have one semi-stable and one asymptotically stable point. When $b = b_1$ $(w_1^I = -6.81)$ the fixed point that is greater than 0.5 is semi-stable, and the other one is asymptotically stable (Fig.5). When $b = b_2$ $(w_1^I = -3.19)$, the fixed point that is smaller than 0.5 is semi-stable, while the other one is asymptotically stable. The semi-stability is equivalent to tangency of line to the curve.

In order to have three fixed points, we choose a value for *b* between b_1 and b_2. In this experiment, we choose $b = (b_1 + b_2)/2 = -0.5$ $(w_1^I = -5)$, the result of which is sketched in Fig. 6. As shown in the figure, the two lateral points are asymptotically stable while the other point is unstable.

For one fixed point case, we should choose $b > b_2$ or $b < b_1$, e.g. $b = -0.271$ $(w_1^I = -2.71)$. It is apparent from Fig. 7 that the fixed point is asymptotically stable.

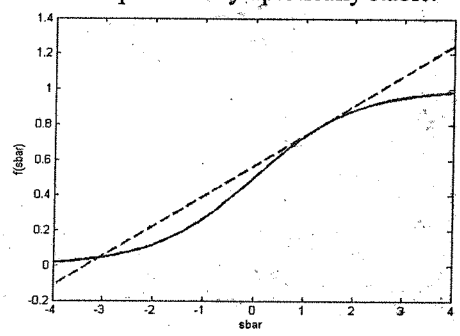


Figure 5: Stability of two fixed points, $b = b_1$ $(w_1^I = -6.81)$.

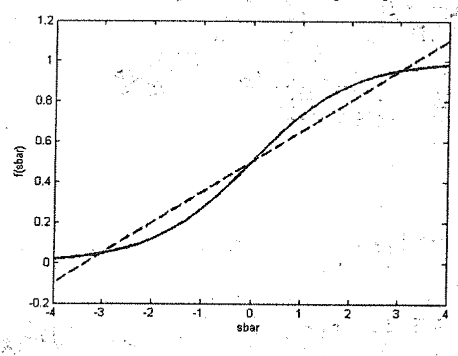


Figure 6: Stability of three fixed points, $b = -0.5$ $(w_1^I = -5)$.

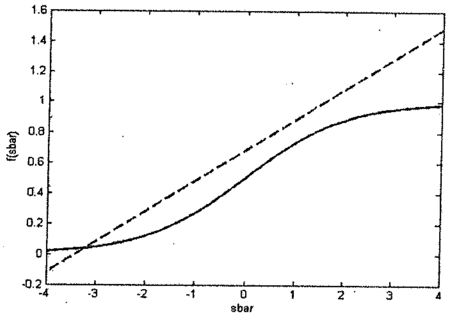


Figure 7: Stability of one fixed point $b = -0.271(w_1^I = -2.71)$

B. Two Self-Feedback Neuron

We have simulated a two-neuron network as illustrated in Fige. 5, for *n=2* and *m=2*. The situation can be formulated as:

TABLE 2

NUMBER OF EQUILIBRIUM POINTS AND THEIR STABILITY IN TWO DIMENSIONAL SPACE

$\left(w_j^D > 4\right)$ Status	b_1	b_2	Stable Points	Unstable Points
1	b_{12}	b_{22}	1	3
2	b_{12}	$>b_{22}$	1	1
3	b_{12}	(b_{21},b_{22})	2	4
4	$>b_{12}$	b_{22}	1	1
5	$>b_{12}$	$>b_{22}$	1	0
6	$>b_{12}$	(b_{21},b_{22})	2	1
7	(b_{11},b_{12})	b_{22}	2	4
8	(b_{11},b_{12})	$>b_{22}$	2	1
9	(b_{11},b_{12})	(b_{21},b_{22})	4	5

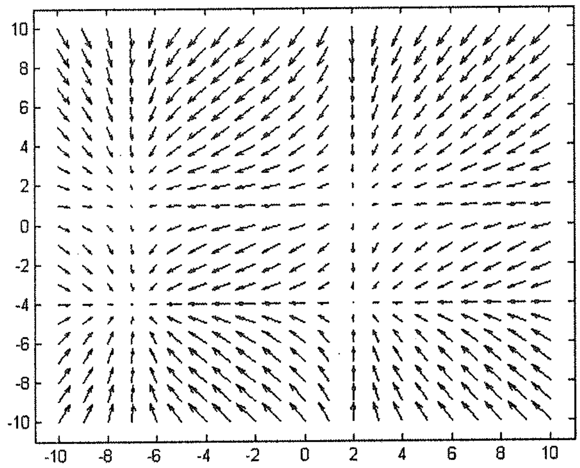
In Fig.5, for *n=2* and *m=2*. The situation can be formulated as:

$$\begin{bmatrix} f(s_1(k)) \\ f(s_2(k)) \end{bmatrix} = \begin{bmatrix} w_{11}^I & w_{12}^I \\ w_{21}^I & w_{22}^I \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix} + \begin{bmatrix} w_1^D & 0 \\ 0 & w_2^D \end{bmatrix} \begin{bmatrix} f(s_1(k-1)) \\ f(s_2(k-1)) \end{bmatrix}$$
$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} f(s_1(k)) \\ f(s_2(k)) \end{bmatrix} \quad (35)$$

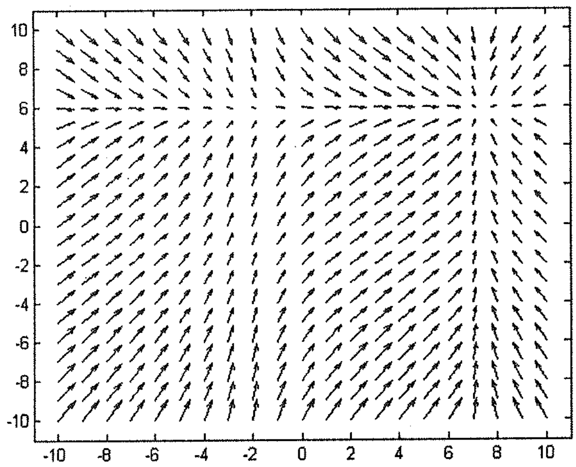
Parameters w_1^D and w_2^D can be determined as indicated in Table 1. In all of the examples shown in Fig.8, we have chosen $w_1^D = 10$ and $w_2^D = 7$. The elements of the input weight matrix will be chosen based on w_1^D and w_2^D, the number of equilibria and their stability. We have summarized the different situations in Table 2, which indeed is an extension of Table 1. For the simulations presented in Fig. 8, the parameter values are given in the caption of the figure.

Suppose that one asymptotically stable equilibrium point is desired. According to Table 2, we have four choices. The major difference between the choices is the number of unstable equilibrium points. For example, at the first row of the table, there are one stable and three unstable equilibrium points; while at the fifth row there is only one equilibrium point which is asymptotically stable. The simulation results for these two examples are shown in Fig. 8(a) and 8(b), respectively. In fact, Fig. 8(a) illustrates the situation in which there are one asymptotically stable and one semi-stable (unstable) fixed point in each dimension. In this case, $p_1 = 2$, $p_2 = 2$, $q_1 = 1$, $q_2 = 1$. Thus, using (33) and (34) we will have: $P = p_1 . p_2 = 4$ and $Q = q_1 . q_2 = 1$. In Fig. 8(b), there is one stable point in each dimension. In this case, $p_1 = 1$, $p_2 = 1$, $q_1 = 1$, $q_2 = 1$. Furthermore, using (33) and (34) we have: $P = p_1 . p_2 = 1$ and $Q = q_1 . q_2 = 1$.

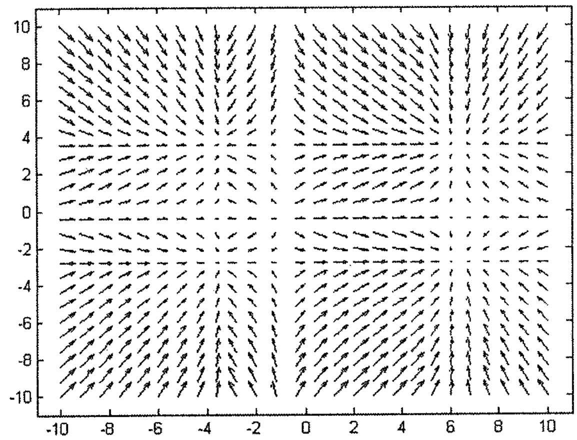
Finally, we can choose the elements of the input weight matrix so that there are three fixed points ($p_1 = p_2 = 3$) in each dimension. Since in this case we have $q_1 = q_2 = 2$, we expect four asymptotically stable equilibrium points ($Q = q_1 . q_2 = 4$). This situation is sketched in Fig. 8c.



(a) $b_1 = b_{11}$, $b_2 = b_{21}$, $w_1^I = -3.405$, $w_2^I = -2.112$



(b) $b_1 = -0.271$, $b_2 = -0.157$, $w_1^I = -1.356$, $w_2^I = -0.555$



(c) $b_1 = -0.373$, $b_2 = -0.459$, $w_1^I = -1.867$, $w_2^I = -1.605$

Figure 7: System trajectory in two dimensional space. Network parameter values are as follows:
$w_1^D = 10, w_2^D = 7, b_{11} = -0.681, b_{12} = -0.319, b_{21} = -0.603, b_{22} = -0.396$

5. Conclusion

Storage of analog pattern vectors with real-valued components using feedback neural networks is of great interest. Theoretically, it was shown how the nature and the number of equilibrium points are affected by weight matrices. In particular, an easy and effective way for selecting the elements of the weight matrices was derived in order to get the number of asymptotically stable equilibrium points under control. Several simulations presented in the paper verified these claims.

6. References

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