Almost Product Structures Ontangent Manifold of a
Space Form

E. Peyghan\textsuperscript{1}, A. Razavi\textsuperscript{2}, A. Heydari\textsuperscript{3}

ABSTRACT

A set of locally product structures on tangent manifold TM of a space form M is pointed out. This is found in a study of a type of Sasaki metric whose second term is a special deformation of the first one. Introducing an adequate almost product structure we find at first a large class of locally almost product structures on TM for a (pseudo)-Riemannian manifold M. When M is a space form, a subset of it is made of locally product structures.

KEYWORDS

Almost product structure, Constant curvature, Nijenhuis tensor, Sasaki metric.

1. INTRODUCTION

Let \( (M, g) \) be a (pseudo)-Riemannian manifold and \( \nabla \) its Levi-Civita connection. In a local chart \((U, (x'))\) we set \( g_{ij} = g(\partial_i, \partial_j) \), where \( \partial_i := \frac{\partial}{\partial x^i} \) and we denote the Christoffel symbols by \( \Gamma^i_{jk}(x) \). Let \( (x', y') \equiv (x, y) \) be the local coordinates on the manifold TM projected on \( M \) by \( \tau \). The indices \( i, j, k, \ldots \) will run from 1 to \( n = \text{dim} M \).

The functions \( N^i_j(x, y) \equiv \Gamma^i_{jk}(x) y^k \) are the local coefficients of a nonlinear connection, that is the local vector fields \( \partial_i = \partial_i - N^i_j(x, y) \partial_j \), where \( \partial_i := \frac{\partial}{\partial y^i} \) span a distribution on TM called horizontal, which is supplementary to the vertical distribution \( u \rightarrow V_u TM = \ker \tau_u, u \in TM \). Let us denote by \( u \rightarrow H_u TM \) the horizontal distribution and let \( (\partial_i, \partial_j) \) be the basis adapted to the decomposition

\[
T_u TM = H_u TM \oplus V_u TM, \quad u \in TM.
\]

The dual basis are \((dx^i, \delta y^j)\) with \( \delta y^j = dy^j + N^i_j(x, y) dx^i \).

The Sasaki metric on TM is as follows:

\[
G_S = g_g(x) dx^i \otimes dx^i + g_g(x) \delta y^i \otimes \delta y^j.
\]

(1.1)

If in the second term of \( G_S \) one replaces \( g_g(x) \) with the components \( h_g(x, y) \) of a generalized Lagrange metric (see Ch.X in [4]) one gets a type of Sasaki metric

\[
G(x, y) = g_g(x) dx^i \otimes dx^i + h_g(x, y) \delta y^i \otimes \delta y^j.
\]

(1.2)

In particular, \( h_g(x, y) \) could be a deformation of \( g_g(x) \), a case studied by M. Anastasici and H. Shimada in [1].

In this paper, we study the metrical structure (1.2) in the case when \( h_g(x, y) \) is the following special deformation of \( g_g(x) \)

\[
h_g(x, y) = a(L^2) g_g(x) + b(L^2) y_i y_j,
\]

(1.3)

where \( L^2 = g_g(x) y^i y^j \), \( y_i = g_g(x) y^i \) and

\textsuperscript{1} E. Peyghan is a Ph.D student of the Department of Mathematics and Computer Science, Amirkabir University of Technology, Tehran, Iran (e-mail: e.peyghan@aut.ac.ir).

\textsuperscript{2} A. Razavi is with the Department of Mathematics and Computer Science, Amirkabir University of Technology, Tehran, Iran (e-mail: arazavi@aut.ac.ir).

\textsuperscript{3} A. Heydari is a Ph.D student of the Department of Mathematics and Computer Science, Amirkabir University of Technology, Tehran, Iran (e-mail: a.heydari@aut.ac.ir).
\[ a, b = \text{Im}(L^2) \subseteq \mathbb{R}, \rightarrow \mathbb{R}, \text{ with } a > 0, b \geq 0. \]

For \( b = 0 \) and \( a = \frac{c^2}{L^2} \) for any constant \( c \), the metrical structure (1.2), (1.3) was studied by R. Miron in [3] as a homogeneous lift of \( g_\phi(x) \) to \( TM \).

In the following section, we introduce an almost product structure which paired with \( G \) given by (1.2), (1.3) which provides a large set of almost product structures on \( TM \).

Finally, we find in section 3 that, when \( (M, g) \) is of constant curvature, some of them are locally product structures.

Let \( P \) be an endomorphism of the tangent bundle \( TM \) satisfying \( P^2 = I \), where \( I = \text{identity} \). Then \( P \) defines an almost product structure on \( M \). If \( g \) is metric on \( M \) such that \( g(PX, PY) = g(X, Y) \) for arbitrary vector fields \( X \) and \( Y \) on \( M \), then the triple \( (M, g, P) \) will introduce the natural almost product structure. If \( P \) is an almost product structure and the Nijenhuis tensor field \( N_P \) of \( P \) vanishes then \( P \) is called a product structure on \( M \).

2. SOME ALMOST PRODUCT STRUCTURES ON \( TM \)

Let \( \tilde{P}_s \) be the almost product structure on \( TM \) given in the adapted basis \((\tilde{e}_i, \tilde{e}_j)\) by

\[ P_s(\tilde{e}_i) = \tilde{e}_j, \quad P_s(\tilde{e}_j) = \tilde{e}_i, \]

\[(\tilde{e}_i, \tilde{e}_j) \]

It is well known that the pair \((G_s, P_s)\) is an almost product structure on \( TM \), that is

\[ G_s(P_sX, P_sY) = G_s(X, Y). \]

We look for a new almost product structure which paired with \( G \) to provide a product structure. We modify \( P_s \) to a linear map \( P \) given in the basis \((\tilde{e}_i, \tilde{e}_j)\) as follows:

\[ P(\tilde{e}_i) = (\alpha \tilde{e}_i + \beta y^k \tilde{e}_k), \]

\[ P(\tilde{e}_j) = (\gamma \tilde{e}_j + \delta y^k \tilde{e}_k), \]

where \( \alpha, \beta, \gamma, \delta \) are functions on \( TM \) to be determined. The condition \( P^2 = I \) leads to

\[ \alpha \gamma = 1, \alpha \beta + \gamma \delta + \beta \delta L^2 = 0. \]

Then the condition \( G(P(X), P(Y)) = G(X, Y) \) gives

\[ a \alpha^2 + 2\alpha \beta \delta + \beta^2 L^2 = b, \]

\[ (2a \alpha \beta + \beta^2 L^2)(a + b L^2) + b \alpha^2 = 0. \]

The solution of the system of equation (2.3), (2.4) is

\[ \alpha = -\frac{1}{\sqrt{\delta}}, \beta = \frac{\sqrt{a + \sqrt{a + b L^2}}}{L^2}, \gamma = -\sqrt{a}, \]

\[ \delta = \frac{\sqrt{a + \sqrt{a + b L^2}}}{L^2}. \]  

(2.5)

We notice that for \( b = 0 \), besides the solution provided by (2.5), that is

\[ \alpha = -\frac{1}{\sqrt{a}}, \gamma = -\sqrt{a}, \beta = \frac{2}{L^2 \sqrt{a}}, \delta = \frac{2 \sqrt{a}}{L^2} \]

(2.6)

There exists also the solution

\[ \alpha = -\frac{1}{\sqrt{a}}, \gamma = -\sqrt{a}, \beta = 0, \delta = 0 \]

(2.7)

Let us make the substitution

\[ a = \frac{a^2}{L^2}, \quad a = \frac{b^2 - a^2}{L^4} \]

Then (2.5) and (2.6) are unified to

\[ \alpha = -\frac{L}{a}, \beta = \frac{a + b}{ab L}, \gamma = -\frac{a}{L}, \delta = \frac{a + b}{L^3}, \]

(2.8)

and (2.7) modifies to

\[ \alpha = -\frac{L}{a}, \gamma = -\frac{a}{L}, \beta = \delta = 0. \]

(2.9)

The metric \( G \) takes the form

\[ G_{ab}(x, y) = g_{ij}(x) dx^i \otimes dx^j \]

\[ + \frac{a^2}{L^2} g_{ij}(x) + \frac{b^2 - a^2}{L^2} y^i y^j \delta y^i \otimes \delta y^j \]

(2.10)

\[ b \geq a > 0. \]

Let \( P_{a,b} \) be the almost product structures given by (2.2), (2.8) and \( P_a \) those given by (2.2), (2.9). Then the pairs \((G_{a,b}, P_{a,b})\) and \((G_{a,a}, P_a)\) are almost product structures on \( TM \).

For \[ a^2 = \frac{L^2}{1 + L^2}, \quad b = L^2 \], the metric \( G_{a,b}(x, y) \) is the Cheeger- Gromoll metric, [5], [6]

\[ G_{CG}(x, y) = g_{ij}(x) dx^i \otimes dx^j \]

\[ + \frac{1}{1 + L^2} (g_{ij}(x) + y_i y_j) \delta y^i \otimes \delta y^j \]

(2.11)

If \[ a^2 = \phi' L^2, \quad b = L^2 (\phi' + 2 \phi'' L^2) \] for \[ \phi: \mathbb{R}, \rightarrow \mathbb{R}, \text{ with } \phi'(t) \neq 0, t \in \text{Im}(L^2) \], one obtains the Antonelli – Hirimiucu metrical structure. [2]
\[ G_{ij}(x,y) = g_{i,j}(x)dx^i \otimes dx^j \]
\[ + (\phi^i g_{i,j}(x) + 2\phi^i y_j y) dy^i \otimes dy^j \]  

(2.12)

3. PRODUCT STRUCTURES ON TM

We know that a Riemannian manifold \( (M, g) \) has a constant curvature \( k \) if

\[ \forall i, j, l, s \quad K_{ijs} = k (g_{is} g_{j} - g_{ij} g_{st}) \]

where \( K_{ijs} \) is the curvature tensor of \( \nabla \). By a contraction with \( g_{ik} \) the Eq. (3.3) reduces to

\[ K_{jst}(x) y^s = \frac{2aL^2 - a}{a^3} (g_{pr} g_{s} - g_{ps} g_{qr}) y^r. \]  

(3.4)

The Eq. (3.4) reminds us of the condition that \( (M, g) \) is of constant curvature (space form). It suggests that we look for functions such that, \[ \frac{2aL^2 - a}{a^3} = k, \] where \( k \) is a constant. For \( t = L^2 \), solving the Bernoulli equation \[ a' = \frac{1}{2t} a + \frac{k}{2t} a^3 \] one gets \( a(L^2) = \sqrt{\frac{L^2}{c - kl^2}} \) for \( c - kl^2 > 0 \), where \( c \) is a constant of integration. So Eq. (3.4) becomes

\[ K_{jst}(x) y^s = k (g_{pr} g_{s} - g_{ps} g_{qr}) y^r, \]

(3.5)

which means that \( (M, g) \) is of constant curvature \( k \). Then we have proved.

**Theorem 3.2.** If the (pseudo)-Riemannian manifold \( (M, g) \) is of constant curvature \( k \in \mathbb{R} \), for \( a(L^2) = \sqrt{\frac{L^2}{c + kl^2}} \)

with \( c \) a constant such that \( c + kl^2 > 0 \), the structures \( (G_{\alpha,\beta}, P_{\alpha}) \) are locally product structures on TM.

The explicit form of these structures are as follows:

\[ G_{\alpha,\beta}(x,y) = g_{i,j}(x) dx^i \otimes dx^j \]
\[ + \left( \frac{1}{c + kl^2} (g_{ij}(x)) \delta y^i \otimes \delta y^j \right) \]

(3.6)

\[ P_{\alpha}(\delta_i, \delta_j) = \sqrt{c + kl^2} \delta_i, \quad P_{\alpha}(\delta_i, \delta_j) = \frac{1}{\sqrt{c + kl^2}} \delta_i \]

(3.7)

**Corollary 3.3.** For \( a(L^2) = c_0 \sqrt{L^2} \), with \( c_0 \) a strict positive constant, the pairs \( (G_{\alpha,\beta}, P_{\alpha}) \) are product structures on TM if and only if \( (M, g) \) is flat.

**Proof.** Since \( a(t) = c_0 \sqrt{t} \) we have
\[ a(t) = \frac{c_0}{2\sqrt{t}} \Rightarrow a'(t^2) = c_0 \]

Therefore, Eq. (3.3) gives \( K_j^k = 0 \), equivalently \( K_{jux}(x) = 0 \). By Theorem 3.2, the structures \((G_{a,x}, P_a)\) are product structures on TM if and only if \((M, g)\) is flat.

Looking at (3.6) and (3.7), we see that the structures \((G_{a,x}, P_a)\) from Corollary 3.3 are very close to \((G_s, P_s)\) which is obtained for \( c=1 \). Thus the Corollary 3.3 covers a well-known result: \((G_s, P_s)\) is a product structure if and only if \((M, g)\) is flat.

4. REFERENCES


