Symmetric Curvature in Lifting Metrics

A. Heydari[i], N. Boroojerdian[ii], E. Peyghan[iii]

ABSTRACT

The symmetric curvature and associated curvatures of a vector bundle E with connection ∇ on a manifold M with connection $\overline{\nabla}$ were introduced. It is well-known that a total space of semi-Riemannian vector bundle over a semi-Riemannian manifold can be made into a semi-Riemannian manifold. In this case, the relation between curvatures of the Levi-Civita connections of E and M was studied. Here, the relation between symmetric curvatures of the Levi-Civita connections and their associated curvatures of E and M is studied.

KEYWORDS

Associated symmetric curvature, Affinewise vector field, Symmetric Lie bracket, Symmetric curvature.

1. PRELIMINARIES

By manifolds we mean C^{∞} real manifolds. The vector bundle (E,π,M,F) will be denoted by

$\pi : E \longrightarrow M,$

with fiber E_p over $p \in M$. VE will denote the vertical bundle of E. It is well known that VE is a subbundle of TE [5]. For $\xi,\eta \in E$ with $\pi(\xi)=\pi(\eta)$ we set $I_{\xi}\eta = \frac{d}{dt}|_{t=0}(\xi + t\eta)$. Clearly $I_{\xi}\eta \in (VE)_{\xi}$, and it is called the vertical lift of η at ξ.

To each connection ∇ on E there corresponds a horizontal subbundle H (of TE), a connection map $k : TE \longrightarrow VE$, and a parallel system P [6]. Let $p \in M$, $u \in T_pM$ and $\xi \in E_p$. There exists a unique vector on H_{ξ} such that its image under π_* is u. This vector is called *the horizontal lift* of u at ξ, and is denoted by $\overline{u}_{\xi}$. The set of all sections of a vector bundle $E \longrightarrow M$ will be denoted by ΓE.

Let E be a Riemannian vector bundle over M. The vector bundles E^* (dual of E), $L(E)\,(=Hom(E,E))$, $\otimes^r E$, $\wedge^r E$ $(1 \le r)$ can be made into Riemannian vector bundles in a natural way.

Let M be a Riemannian manifold, a submanifold N of M is also Riemannian manifold. Let ∇^M and ∇^N denote the Levi-Civita connections of M and N, respectively, and E be the restriction of TM on N (or equivalently, E be the pull-back of TM over the inclusion map $i : N \longrightarrow M$). The pull-back of ∇^M, which is a connection on E will be denoted by the same symbol ∇^M. Let $p_1 : E \longrightarrow TN$ be the orthogonal projection. Then for each $U,V \in X(N) \subseteq \Gamma E$

$$\nabla^N_U V = p_1(\nabla^M_U V).$$

Let $TN^{\perp}$ be the orthogonal complement of the vector bundle TN in E, and $p_2 : E \longrightarrow TN^{\perp}$ be the orthogonal projection. The map $\pi : TN \otimes TN \longrightarrow TN^{\perp}$ which is defined by

$$\pi(U,V) = p_2(\nabla^M_U V) = \nabla^M_U V - \nabla^N_U V,$$

is a symmetric tensor, called *second fundamental form*

i A. Heydari is a Ph.D student of the Department of Mathematics and computer science, Amirkabir University of Technology, Tehran, Iran (e-mail: a_heydari@aut.ac.ir).

ii N. Borojerdian is with the Department of Mathematics and computer science, Amirkabir University of Technology, Tehran, Iran (e-mail: broojerd@aut.ac.ir).

iii E. Peyghan is a Ph.D student of the Department of Mathematics and computer science, Amirkabir University of Technology, Tehran, Iran (e-mail: e_peyghani@aut.ac.ir).

of N [4].

A. The symmetric curvature tensor [3]

Let $\bar{\nabla}$ be a torsion free connection on M. Since $2\bar{\nabla}_U V$ is a bilinear map with respect to vector fields U and V, it can be written as the sum of its symmetric and antisymmetric parts as follows

$$2\bar{\nabla}_U V = (\bar{\nabla}_U V + \bar{\nabla}_V U) + (\bar{\nabla}_U V - \bar{\nabla}_V U)$$
$$= \bar{\nabla}_U V + \bar{\nabla}_V U + [U,V].$$

The symmetric bracket of two vector fields U and V on M is defined and denoted by

$$[U,V]^s = \bar{\nabla}_U V + \bar{\nabla}_V U.$$

For $U,V \in X(M)$ and $f \in C^\infty(M)$, we have

$$[fU,V]^s = f[U,V]^s + V(f)U.$$

Definition 1. Let $\bar{\nabla}$ be a linear connection on M. A vector field U is called a *geodesic vector field* if its integral curves are geodesics.

Locally, geodesic vector fields exist on any manifold. In fact, for every point $p \in M$ and $v \in T_pM$ there exists a local geodesic vector field U that is defined on a neighborhood of p in which $U_p = v$. A vector field U is a geodesic field if and only if $[U,U]^s = 2\bar{\nabla}_U U = 0$.

Let E be a vector bundle with the connection ∇ over M and let $\bar{\nabla}$ be a torsion-free linear connection on M. For every section $Z \in \Gamma(E)$ the bilinear map

$$\nabla\nabla Z : X(M) \times X(M) \longrightarrow \Gamma(E),$$

defined by

$$\nabla\nabla Z(U,V) = \nabla_U \nabla_V Z - \nabla_{\bar{\nabla}_U V} Z,$$

can be written as the sum of its symmetric and antisymmetric parts as follows:

$$\nabla\nabla Z(U,V) =$$
$$\frac{1}{2}(\nabla_U \nabla_V Z + \nabla_V \nabla_U Z - \nabla_{\bar{\nabla}_U V} Z - \nabla_{\bar{\nabla}_V U} Z) +$$
$$\frac{1}{2}(\nabla_U \nabla_V Z - \nabla_V \nabla_U Z - \nabla_{\bar{\nabla}_U V} Z + \nabla_{\bar{\nabla}_V U} Z)$$
$$= \frac{1}{2}(\nabla_U \nabla_V Z + \nabla_V \nabla_U Z - \nabla_{[U,V]^s} Z) +$$
$$\frac{1}{2}(\nabla_U \nabla_V Z - \nabla_V \nabla_U Z - \nabla_{[U,V]} Z).$$

The last expression in the parentheses is the antisymmetric part of $\nabla\nabla Z$ and is the curvature of ∇ which is denoted by $R(U,V)Z$. The first expression is the symmetric part of $\nabla\nabla Z$ and we call it the symmetric curvature of ∇ and denote it by $R_Z^s(U,V)$, so

$$R_Z^s(U,V) = \nabla_U \nabla_V Z + \nabla_V \nabla_U Z - \nabla_{[U,V]^s} Z.$$

Note that $R_Z^s(U,V)$ is not tensorial in argument Z. But it is tensorial and symmetric in two arguments U,V. The curvature tensor R does not depend on the choice of $\bar{\nabla}$, but R^s does depend on $\bar{\nabla}$.

For geodesic vector fields we have a simple relation for computing the symmetric curvature. If U is a geodesic vector field on M, then for any section Z of E, we have

$$R_Z^s(U,U) = 2\nabla_U \nabla_U Z.$$

Definition 2. Let ∇ be a connection on a vector bundle $E \longrightarrow M$ and $\bar{\nabla}$ be a torsion free connection on M. A section $Z \in \Gamma E$ is called *affinewise* if its symmetric curvature tensor vanishes, i.e., $R_Z^s = 0$. In particular, an affinewise section of $E = TM$ is called affinewise vector field.

The set of affinewise sections is a linear subspace of ΓE. In particular, the zero section is affinewise.

Example 1. Let Z be a parallel section of vector bundle E. Since for every vector field V, $\nabla_V Z = 0$, we find $R_Z^s = 0$. Thus, all parallel sections are affinewise.

Example 2. Consider a trivial vector bundle $E = R^n \times V$ with the trivial connection on it. A section of E is a smooth map $Z : R^n \longrightarrow V$. By a routine calculation we find $R_Z^s = 0$ if and only if Z is an affine map. So affinewise sections of E are the same as affine maps.

Knowing the second fundamental form π of N, we compute symmetric curvatures of N in terms of the corresponding curvatures of M. For $U,V \in X(N)$ let $[U,V]_M^s, [U,V]_N^s$ be symmetric brackets on M and N, respectively. Then

$$[U,V]_N^s = p_1([U,V]_M^s)$$
$$2\pi(U,V) = [U,V]_M^s - [U,V]_N^s$$

Lemma 1. Let R^s and $\hat{R}^s$ denote the symmetric curvature tensors of M and N, respectively. Let $W' \in X(M)$ and $W \in X(N)$ be vector fields such that W' is equal to W on N. Then for each $U,V,P \in X(N)$ we have

$$< \hat{R}^s_W(U,V),P >=< \hat{R}^s_{W'}(U,V),P > + 2 < \nabla^M_{\pi(U,V)} W',P > + < \pi(U,P),\pi(V,W) > + < \pi(V,P),\pi(U,W) > .$$

Proof. Let U be a geodesic vector field on N, then $[U,U]^s_N = 0$. So, $[U,U]^s_M = 2\pi(U,U)$, and we have:

$$\begin{aligned}
< \hat{R}^s_W(U,U),P > &= < 2\nabla^N_U \nabla^N_U W,P > \\
&= < 2p_1(\nabla^M_U \nabla^N_U W),P > \\
&= < 2\nabla^M_U (\nabla^M_U W - \pi(U,W)),P > \\
&= < 2\nabla^M_U \nabla^M_U W,P > - < 2\nabla^M_U \pi(U,W),P > \\
&= < 2\nabla^M_U \nabla^M_U W - \nabla^M_{[U,U]^s_M} W',P > + \\
&\quad < \nabla^M_{[U,U]^s_M} W',P > - < 2\nabla^M_U \pi(U,W),P > \\
&= < R^s_{W'}(U,U),P > + < \nabla^M_{[U,U]^s_M} W',P > - \\
&\quad 2U < \pi(U,W),P > + 2 < \pi(U,W),\nabla^M_U P > \\
&= < R^s_{W'}(U,U),P > + < \nabla^M_{[U,U]^s_M} W',P > + \\
&\quad 2 < \pi(U,W),p_2(\nabla^M_U P) > \\
&= < R^s_{W'}(U,U),P > + 2 < \nabla^M_{\pi(U,U)} W',P > + \\
&\quad 2 < \pi(U,P),\pi(U,W) >
\end{aligned}$$

Since R^s is symmetric with respect to the first and second components, proof is complete.

B.Fundamental vector fields of a vector bundle

Assume that $\pi : E \longrightarrow M$ is vector bundle. A map $F : E \longrightarrow E$ is called a strong bundle map if every fiber $E_p (p \in M)$ is invariant under F .If restriction of F to each E_p is linear, it is called a linear strong bundle map.

To each strong bundle map $F : E \longrightarrow E$ (not necessarily linear) there corresponds a vertical vector field of E (a section of VE) which will be denoted by $\tilde{F}$ and is defined by

$$\tilde{F}_\xi = I_\xi F(\xi) , \quad \xi \in E .$$

$\tilde{F}$ *is smooth.* [1]

For example, if $F = 1_E$, then $\tilde{1}_E$ is the radial vector field on E. The set of all vertical vector fields on E as well as the set of all strong bundle maps on E, are modules over $C^\infty(E)$. From the definition of $\tilde{F}$ and the local representations of F and $\tilde{F}$, we see that the map $F \mapsto \tilde{F}$ is a linear isomorphism between the above modules.

Let ∇ be a connection on E throughout the paper. To each strong bundle map $A : E \longrightarrow TM$ (not necessarily linear) there corresponds a horizontal vector field on E (a section of H) which will be denoted by $\overline{A}$ and is defined by

$$\overline{A}_\xi = \overline{A(\xi)}_\xi .$$

$\overline{A}_\xi$ is smooth. [1]

For example if $E = TM$ and $A = 1_{TM}$, then $\overline{1_{TM}}$ is the geodesic spray of ∇. The set of all horizontal vector fields on E as well as the set of all strong bundle maps from E to TM are modules over $C^\infty(E)$.

From the definition of $\overline{A}$ and the local representations of A and $\overline{A}$ it is clear that the map $A \mapsto \overline{A}$ is a linear isomorphism between these modules.

For each $X \in \Gamma E$ (resp. $U \in X(M)$) $X \circ \pi$ (resp. $\overline{U \circ \pi}$) is called *vertical lift* of X (resp. the horizontal lift of U) and it is denoted by IX (resp. $\overline{U}$).

Proposition 1. [1] Let $F : E \longrightarrow E$ and $A : E \longrightarrow TM$ be linear strong bundle maps, and R be the curvature tensor of ∇ then for $X,Y \in \Gamma E$ and $U,V \in X(M)$ we have

$$[IX, IY] = 0 \tag{1}$$

$$[\overline{U}, IX] = I\nabla_U X \tag{2}$$

$$[\overline{U},\overline{V}] = \overline{[U,V]} - \widetilde{R(U,V)} \tag{3}$$

$$[IX, \tilde{F}] = IF \circ X \tag{4}$$

$$[\overline{U}, \tilde{F}] = \widetilde{\nabla_U F} \tag{5}$$

$$[IX, \overline{A}] = \overline{A \circ X - \nabla_{A(.)} X} \tag{6}$$

$$[\overline{U}, \overline{A}] = \overline{L_U A} - \widetilde{R(U,A(.))(.)} \tag{7}$$

C.Lift of Riemannian metrics

Let E be a Riemannian vector bundle, and M be a Riemannian manifold and ∇ be a connection on E. We can lift the metric of M to E as follows:

$$\hat{u},\hat{v} \in T_\xi E , \quad < \hat{u},\hat{v} > = < k(\hat{u}), k(\hat{v}) >_E + < d\pi(\hat{u}), d\pi(\hat{v}) >_M$$

Thus E becomes a Riemannian manifold. At each point $\xi \in E$, the horizontal space H_ξ and the vertical space $(VE)_\xi$ are orthogonal to each other, and inner product on H_ξ and $(VE)_\xi$ are the same as the inner products on $T_{\pi(\xi)}M$ and $E_{\pi(\xi)}$ under the isomorphisms

$\pi_* : H_\xi \longrightarrow T_{\pi(\xi)}M$ and $k : (VE)_\xi \longrightarrow E_{\pi(\xi)}$, respectively. So scalar products of horizontal and vertical vector fields of E are zero.

From now on, we assume that the metric of the vector bundle is parallel with respect to ∇, namely for every $X,Y \in \Gamma E$ and $U \in X(M)$ we have

$$U < X,Y > = < \nabla_U X, Y > + < X, \nabla_U Y > .$$

D. The Levi-Civita connection of E

Let $\blacklozenge(E)$ be the vector bundle over M, whose fiber at each point $p \in M$ is $\blacklozenge(E_p)$ and let $L(\wedge^2 TM, \blacklozenge(E))$ be the vector bundle over M, whose fiber at each point $p \in M$ is $L(\wedge^2 T_p M, \blacklozenge(E_p))$ (space of linear maps between these vector spaces). Then R (the curvature tensor of ∇) is a section of $L(\wedge^2 TM, \blacklozenge(E))$. As mentioned above, $\blacklozenge(E)$ and $\blacklozenge(TM)$ are naturally isomorphic to $\wedge^2 E$ and $\wedge^2 TM$. So we use them interchangeably, and assume that

$$R \in \Gamma L(\wedge^2 TM, \wedge^2 E).$$

Then

$$R^* \in \Gamma L(\wedge^2 E, \wedge^2 TM).$$

or

$$R^* \in \Gamma L(\wedge^2 E, \blacklozenge(TM)).$$

which is defined explicitly and uniquely by the following formula

$$< R(U,V)(X),Y >_E = < R^*(X,Y)(U),V >_M$$

where $X,Y \in \Gamma E, U,V \in X(M)$.

For example if $E = TM$, and $\nabla = \nabla^M$ (the Levi-Civita connection of M), then $R^* = R$. In other words, R is symmetric with respect to the inner product of $\wedge^2 TM$.

Theorem 1. [1] Let $\overline{\nabla}$ denote the Levi-Civita connection of E. If $F : E \longrightarrow E$ and $A : E \longrightarrow TM$ are linear strong bundle maps and $X,Y \in \Gamma E, U,V \in X(M)$, then

$$\overline{\nabla}_{IX} IY = 0, \quad (8)$$

$$\overline{\nabla}_{\overline{U}} \overline{V} = \overline{\nabla^M_U V} - \frac{1}{2}\widetilde{R(U,V)}, \quad (9)$$

$$\overline{\nabla}_{IX} \overline{U} = \frac{1}{2}\overline{R^*(.,X)(U)}, \quad (10)$$

$$\overline{\nabla}_{IX} \tilde{F} = IF \circ X, \quad (11)$$

$$\overline{\nabla}_{IX} \overline{A} = \overline{A \circ X} + \frac{1}{2}\overline{R^*(.,X)(A(.))}, \quad (12)$$

$$\overline{\nabla}_{\overline{U}} \tilde{F} = \widetilde{\nabla_U F} + \frac{1}{2}\overline{R^*(.,F(.))(U)}, \quad (13)$$

$$\overline{\nabla}_{\overline{U}} \overline{A} = \overline{\nabla_U A} - \frac{1}{2}\widetilde{R(U,A(.))(.)}. \quad (14)$$

Theorem 2. [1] Let $F : E \longrightarrow E$ and $A : E \longrightarrow TM$ be linear strong bundle maps, and R be the curvature tensor of ∇ then for $X,Y \in \Gamma E$ and $U,V \in X(M)$ we have

$$[IX, IY]^s = 0 \quad (15)$$

$$[IX, \overline{U}]^s = \overline{R^*(.,X)(U)} + I\nabla_U X \quad (16)$$

$$[\overline{U}, \overline{V}]^s = \overline{[U,V]^s} \quad (17)$$

$$[IX, \tilde{F}]^s = IF \circ X \quad (18)$$

$$[\overline{U}, \tilde{F}]^s = \widetilde{\nabla_U F} + \overline{R^*(.,F(.))(U)} \quad (19)$$

$$[IX, \overline{A}]^s = \overline{A \circ X} + \overline{R^*(.,X)A(.)} + \overline{\nabla_{A(.)} X} \quad (20)$$

$$[\overline{U}, \overline{A}]^s = \overline{L^s_U A}, \quad (21)$$

where L^s_U is defined by $L^s_U = 2\nabla_U - L_U$. [3]

Proof. The proof is by direct computation and using Proposition 1 and Theorem 1. We compute relations (17) and (20).

From definition of symmetric Lie bracket, we have

$$[\overline{U}, \overline{V}]^s = 2\overline{\nabla}_{\overline{U}} \overline{V} - [\overline{U}, \overline{V}].$$

By putting relations (3) and (9) in above equation conclude that

$$[\overline{U}, \overline{V}]^s = 2(\overline{\nabla^M_U V} - \frac{1}{2}\widetilde{R(U,V)}) - \overline{[U,V]} + \widetilde{R(U,V)}$$
$$= 2\overline{\nabla^M_U V} - \overline{[U,V]} = \overline{[U,V]^s}.$$

Similar to above computation and by using relations (6) and (12) we have

$$[IX, \overline{A}]^s = 2\overline{\nabla}_{IX} \overline{A} - [IX, \overline{A}]$$
$$= 2(\overline{A \circ X} + \frac{1}{2}\overline{R^*(\cdot,X)(A(\cdot))}) - (\overline{A \circ X} - \overline{\nabla_{A(\cdot)} X})$$
$$= \overline{A \circ X} + \overline{R^*(\cdot,X)(A(\cdot))} + \overline{\nabla_{A(\cdot)} X}.$$

2. SYMMETRIC CURVATURE TENSOR OF E

Theorem 2. Let $\overline{R^s}$, R^s and $R^{s,M}$ denote the symmetric curvature tensors of $\overline{\nabla}$, ∇ and ∇^M, respectively. Assume that $X,Y,Z \in \Gamma E$ and

$U,V,W \in X(M)$. Then

$$\overline{R^s}_{IZ}(IX,IY)=0, \qquad (22)$$

$$\overline{R^s}_{IY}(\overline{U},IX)=\frac{1}{2}\overline{R^*(X,Y)(U)}+\frac{1}{4}\overline{R^*(\cdot,X)(R^*(\cdot,Y)(U))}$$

$$-\frac{1}{2}\overline{R^*(\cdot,Y)(R^*(\cdot,X)(U))}-\overline{\nabla_{R^*(\cdot,X)(U)}Y}, \qquad (23)$$

$$\overline{R^s}_{\overline{U}}(IX,IY)=\frac{1}{4}\overline{R^*(\cdot,X)(R^*(\cdot,Y)(U))}+\frac{1}{4}\overline{R^*(\cdot,Y)(R^*(\cdot,X)(U))}, \qquad (24)$$

$$\overline{R^s}_{\overline{V}}(\overline{U},IX)=\frac{1}{2}\overline{\nabla_U R^*(\cdot,X)(V)}-\overline{\nabla_V R^*(\cdot,X)(U)}$$
$$-\frac{1}{4}\overline{R(U,R^*(\cdot,X)(V))(\cdot)}-\frac{1}{2}\overline{R(V,R^*(\cdot,X)(U))(\cdot)}$$
$$+\frac{1}{2}\overline{R^*(\cdot,X)(\nabla^M_U V)}-\frac{1}{2}I(R(U,V)(X))+$$
$$\overline{L_V R^*(\cdot,X)(U)}-\frac{1}{2}\overline{R^*(\cdot,\nabla_U X)(V)}, \qquad (25)$$

$$\overline{R^s}_{IX}(\overline{U},\overline{V})=IR^s_X(U,V)+\frac{1}{2}\overline{\nabla_U R^*(\cdot,X)(V)}+$$
$$\frac{1}{2}\overline{\nabla_V R^*(\cdot,X)(U)}-\frac{1}{4}\overline{R(U,R^*(\cdot,X)(V))(\cdot)}-$$
$$\frac{1}{4}\overline{R(V,R^*(\cdot,X)(U))(\cdot)}+\frac{1}{2}\overline{R^*(\cdot,\nabla_V X)(U)}+$$
$$\frac{1}{2}\overline{R^*(\cdot,\nabla_U X)(V)}-\frac{1}{2}\overline{R^*(\cdot,X)([U,V]^s)} \qquad (26)$$

$$\overline{R^s}_{\overline{W}}(\overline{U},\overline{V})=\overline{R^{s,M}_W(U,V)}-\frac{1}{2}\overline{(\nabla_U R)(V,W)}-$$
$$\frac{1}{2}\overline{(\nabla_V R)(U,W)}-\frac{1}{4}\overline{R^*(\cdot,R(V,W)(\cdot))(U)}-$$
$$\frac{1}{4}\overline{R^*(\cdot,R(U,W)(\cdot))(V)}-\overline{R(V,\nabla_U W)}-$$
$$\overline{R(U,\nabla_V W)}. \qquad (27)$$

Proof. The proof is by direct computation. We compute the relation (27).

From definition of symmetric curvature tensor we have

$$\overline{R^s}_{\overline{W}}(\overline{U},\overline{V})=\overline{\nabla}_{\overline{U}}\overline{\nabla}_{\overline{V}}\overline{W}+\overline{\nabla}_{\overline{V}}\overline{\nabla}_{\overline{U}}\overline{W}-\overline{\nabla}_{[\overline{U},\overline{V}]^s}\overline{W}.(*)$$

From equations (9), (13) and direct computation we get

$$\overline{\nabla}_{\overline{U}}\overline{\nabla}_{\overline{V}}\overline{W}=\overline{\nabla^M_U\nabla^M_V W}-\frac{1}{2}\overline{R(U,\nabla^M_V W)}-\frac{1}{2}\overline{\nabla_U R(V,W)}-\frac{1}{4}\overline{R^*(\cdot,R(V,W)(\cdot))(U)}.$$

By interchanging of U and V in above equation we obtain $\overline{\nabla}_{\overline{V}}\overline{\nabla}_{\overline{U}}\overline{W}$. Also we have

$$\overline{\nabla}_{[\overline{U},\overline{V}]^s}\overline{W}=\overline{\nabla}_{\overline{[U,V]^s}}\overline{W}=\overline{\nabla^M_{[U,V]^s}W}-\frac{1}{2}\overline{R([U,V]^s,W)}.$$

By putting the above relations in $(*)$, *the proof of theorem is complete.*

Corollary 1. If the curvature of ∇ vanishes, i.e., $R=0$, then

$$\overline{R^s}_{IZ}(IX,IY)=\overline{R^s}_{IZ}(\overline{U},IX)=0,$$
$$\overline{R^s}_{IZ}(\overline{U},\overline{V})=IR^s_Z(U,V),$$
$$\overline{R^s}_{\overline{W}}(\overline{U},\overline{V})=\overline{R^{s,M}_W(U,V)}.$$

Corollary 2. Let Z be a parallel section of E. Then IZ is an affinewise vector field on E.

Proof. Since Z is parallel, then $R(U,V)Z=0$ for all $U,V\in X(M)$. Therefore we have

$$<R^*(X,Z)(U),V>=<R(U,V)X,Z>=0$$

where $X\in \Gamma E$, $U,V\in X(M)$.

From above equation, we conclude $R^*(\cdot,Z)=0$. Thus, from relations (22), (23) and (24) we have $\overline{R^s}_{IZ}(IX,IY)=0$, $\overline{R^s}_{IZ}(IX,\overline{U})=0$ and $\overline{R^s}_{IZ}(\overline{U},\overline{V})=0$. So, IZ is an affinewise vector field.

Corollary 3. If IZ is an affinewise vector field on E, then Z is an affinewise section of E. The converse is true if $R=0$.

Proof. Let IZ is affinewise, From relations $\overline{R^s}_{IZ}(IX,IY)=\overline{R^s}_{IZ}(IX,\overline{U})=\overline{R^s}_{IZ}(\overline{U},\overline{V})=0$, following equations is holds

$$R^*(X,Z)(U)+\frac{1}{2}R^*(\cdot,X)(R^*(\cdot,Z)(U))-R^*(\cdot,Z)(R^*(\cdot,X)(U))=0, \qquad (28)$$

$$\nabla_{R^*(\cdot,X)(U)} Z = 0, \quad (29)$$

$$\nabla_U R^*(\cdot,Z)(V) + \nabla_V R^*(\cdot,Z)(U) + R^*(\cdot,\nabla_V Z)(U) + R^*(\cdot,\nabla_U Z)(V) - R^*(\cdot,Z)([U,V]^s) = 0, \quad (30)$$

$$IR_Z^s(U,V) - \frac{1}{4}\{\overline{R(U,R^*(\cdot,Z)(V))(\cdot)} + \overline{R(V,R^*(\cdot,Z)(U))(\cdot)}\} = 0 \quad (31)$$

Setting $X = Z$,we can conclude that

$R^*(\cdot,Z)(R^*(\cdot,Z)(U)) = 0$.

For all $Y \in \Gamma E$, we get

$$< R^*(Y,Z)(U), R^*(Y,Z)(U) >= < R^*(Y,Z)(R^*(Y,Z)(U)),U >= 0.$$

So,

$R^*(\cdot,Z)(U) = 0$.

From the above equation and (31) we find $IR_Z^s(U,V) = 0$, so Z is affinewise.

Conversely, if Z is affinewise and $R = 0$, then $R^* = 0$. Therefore, from relations (22),(23) and (26) we can conclude that $\overline{R^s}_{IZ}(\cdot,\cdot) = 0$, i.e., IZ is affinewise.

Corollary 4. If $\overline{U}$ is an affinewise vector field, then U is affinewise vector field. Conversely, if U is an affinewise vector field and $R = 0$, then $\overline{U}$ is affinewise.

Proof. Let $\overline{U}$ be affinewise, so the following relations hold:

$$R^*(\cdot,X)(R^*(\cdot,Y)(U)) + R^*(\cdot,Y)(R^*(\cdot,X)(U)) = 0, \quad (32)$$

$$R^*(\cdot,X)(\nabla_V^M U) + \nabla_V R^*(\cdot,X)(U) - 2\nabla_U R^*(\cdot,X)(V) - R^*(\cdot,\nabla_V X)(U) + 2L_U R^*(\cdot,X)(V) = 0, \quad (33)$$

$$2I(R(V,U)(X)) + 2\overline{R(U,(R^*(.,X)(V)))(\cdot)} + \overline{R(V,(R^*(.,X)(U)))(\cdot)} = 0, \quad (34)$$

$$4R_U^{s,M}(V,W) - R^*(\cdot,R(W,U)(\cdot))(V) - R^*(\cdot,R(V,U)(\cdot))(W) = 0, \quad (35)$$

$$(\nabla_V R)(W,U) + (\nabla_W R)(V,U) + 2R(W,\nabla_V U) + 2R(V,\nabla_W U) = 0. \quad (36)$$

We can set $Y = X$ in the relation (32), and obtain

$R^*(\cdot,X)(R^*(\cdot,X)(U)) = 0,$

so

$< R^*(\cdot,X)(R^*(\cdot,X)(U)),U >= 0,$

and then

$< R^*(\cdot,X)(U), R^*(\cdot,X)(U) >= 0.$

It means that $R^*(\cdot,X)(U) = 0$. By definition of R^* it holds if and only if $R(\cdot,U)(X) = 0$. From (35) we find

$\forall V,W \in X(M) \quad R_U^s(V,W) = 0.$

So, U is an affine wise vector field.

Conversely if $R = 0$ and U is an affinewise then by Corollary1 we conclude that $\overline{U}$ is affinewise.

3. ASSOCIATED CURVATURES

By contracting symmetric curvature we can find new curvatures. This contraction can be done in two ways.

Definition 3. [4] Let $\{U_i\}$ be a basis of local vector fields on M with the dual basis $\{\omega^i\}$. For every vector field Z, we assign a 1-form ω_Z as follows and we call it the *form curvature along* Z:

$$\omega_Z(U) = \sum_i \omega^i(R_Z^s(U_i,U)).$$

Theorem 4. Let $\{U_i\}$ be an orthonormal basis of local vector fields on Riemannian manifold M and suppose $\{X_j\}$ are orthogonal local sections for Riemannian bundle $\pi : E \longrightarrow M$. Then we have

$$\omega_{IZ}(\overline{V}) = \frac{1}{2}\sum_i \{<\overline{U_i}, \overline{\nabla_{U_i} R^*(\cdot,Z)(V)}> + <\overline{U_i}, \overline{\nabla_V R^*(\cdot,Z)(U_i)}> + <\overline{U_i}, \overline{R^*(\cdot,\nabla_{U_i} Z)(V)}> - <\overline{U_i}, \overline{R^*(\cdot,Z)([U_i,V]^s)}>\} - \sum_j < IX_j, \overline{\nabla_{R^*(\cdot,X_j)(V)} Z} >,$$

$$\omega_{IZ}(IY) = \frac{1}{4}\sum_i < R^*(\cdot,Z)(U_i), R^*(\cdot,Y)(U_i) >$$

$$\omega_{\overline{W}}(\overline{V}) = \overline{\omega_W(V)} + \frac{1}{4}\sum_i < R(V,U_i)(.), R(W,U_i)(.) >$$

$$-\frac{3}{4}\sum_j < R^*(\cdot,X_j)(V), R^*(\cdot,X_j)(W) >$$

$$\omega_{\overline{W}}(IY) = \frac{1}{2}\sum_i \{<\overline{U_i}, \overline{\nabla_{U_i} R^*(\cdot,Y)(W)}> -$$

$$2<\overline{U_i}, \overline{\nabla_W R^*(\cdot,Y)(U_i)}> +$$

$$<\overline{U_i}, \overline{R^*(\cdot,Y)(\nabla_{U_i} W)}> + 2<\overline{U_i}, \overline{L_W R^*(\cdot,Y)(U_i)}>$$

$$-<\overline{U_i}, \overline{R^*(\cdot,\nabla_{U_i} Y)(W)}>\}.$$

Proof. From definition of ω_{IZ} we have

$$\omega_{IZ}(\cdot) = \sum_i <\overline{U_i}, \overline{R^s}_{IZ}(\overline{U_i},\cdot)> + \sum_j <IX_j, \overline{R^s}_{IZ}(IX_j,\cdot)>.$$

So, by using formulas $(22)-(27)$ the proof can be done.

Definition 4. [4] Let $\{U_i\}$ be an orthonormal basis of local vector fields on the Riemannian manifold M. For every vector field Z, we assign a vector field X_Z as follows and we call it *the vector curvature along* Z:

$$X_Z = \sum_i R_Z^s(U_i, U_i).$$

Theorem 5. Let $\{U_i\}$ be a orthonormal basis of local vector fields on a Riemannian manifold M and suppose $\{X_j\}$ orthogonal local sections for Riemannian bundle $\pi: E \longrightarrow M$. Then we have

$$X_{IZ} = \sum_i \{IR_Z^s(U_i,U_i) + \overline{\nabla_{U_i} R^*(\cdot,Z)(U_i)} -$$

$$\frac{1}{2}\overline{R(U_i, R^*(\cdot,Z)(U_i))(\cdot)} + \overline{R^*(\cdot,\nabla_{U_i} Z)(U_i)} -$$

$$\frac{1}{2}\overline{R^*(\cdot,Z)([U_i,U_i]^s)}\}, \qquad (37)$$

$$X_{\overline{W}} = \sum_i \{\overline{X_W} - \overline{(\nabla_{U_i} R)(U_i,W)} -$$

$$2\overline{R(U_i,\nabla_{U_i} W)} - \frac{1}{2}\overline{R^*(\cdot,R(U_i,W)(\cdot))(U_i)}\} +$$

$$\frac{1}{2}\sum_j \overline{R^*(\cdot,X_j)(R^*(\cdot,X_j)(W))} \qquad (38)$$

Proof. From definition of X_{IZ} we have

$$X_{IZ} = \sum \overline{R^s}_{IZ}(\overline{U_i},\overline{U_i}) + \sum \overline{R^s}_{IZ}(IX_j, IX_j).$$

By (22), the second statement of above relation is zero. Then (37) is hold from (26). Similarly, other equation can be proved.

4. References

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