# *The General Response Formula and Global Stability Of the Generalized Wave Model*

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*ABSTRACT*

The generalized wave model (GWM) representation of singular 2-D systems has been recently introduced [1]. Based on this index-dependent Model, the general response formula is derived to express the global state of the model solely as a function of inputs and boundary conditions within the admissible subspace. The notion of global asymptotic stability of the GWM is defined and sufficient conditions for stability of the model are obtained which guarantee local stability of the original singular 2-D model.

*KEY WORDS:*

Singular 2-D System, Generalized Wave Model, General Response Formula, Stability.

## 1. Introduction

Interest in the theory and practice of 2-D (two dimensional) and the more general m-D systems grew heavily during the last two to three decades due to the variety of applications in areas such as image processing, pattern recognition, distributed parameter systems, models of multipass processes, and numerical analysis of partial differential equations.

The most popular state-space 2-D models were introduced by Givone-Roesser (GR) [2], and Fornasini-Marchesini (FM) [3] and have proven useful in representing 2-D linear systems. However, in the 2-D plane there is no natural notion of causality. Indeed, only partial orderings may be defined for the double index set $(i,j)$ with local state and input vectors in the first quadrant. This has led to some awkwardness in extending results from the 1-D state space theories to their 2-D counterparts.

For this reason, Porter and Aravena [4] introduced the wave advanced model (WAM) to cast 2-D system under consideration as a one dimensional non-square, variable structure model. In fact, augmentation of local states of 2-D models with equal summation of indices $i+j$ as points, form global states of WAM as lines with the slope of -1 in the first quadrant.

While thinking along these lines, the hyperbolic equation and the heat equation, which are two variable partial differential equations with boundary conditions specified along all sides of a planar region, or non-recursible masks in image processing applications, cannot be represented by state-space models with such boundaries. On the other hand, the singular 2-D models, namely, singular Roesser model (SRM) [5] and singular modified Fornasini-Marchesini model (SMFM) [6], are more natural to describe such processes due to their generality and their ability to express multi-directional dynamic and algebraic relationships among the system states [7].

We consider the singular modified Fornasini-Marchesini (SMFM) model as follows.

$$Ex(i+1,j+1) = A_1 x(i+1,j) + A_2 x(i,j+1) + B_1 u(i+1,j) + B_2 u(i,j+1) \qquad (1)$$

with $i$ and $j$ both non-negative integer-valued vertical and horizontal coordinates respectively, $x(i,j) \in \Re^{r}$ and $u(i,j) \in \Re^{m}$ as local state and input vectors respectively and matrices of appropriate dimensions with $E$ possibly singular. Boundary conditions for (1) are given by:

$$x(i,0) \ \ and \ \ x(0,j) \qquad for \ \ i,j \in Z^{+}. \qquad (2)$$

It is well known [8] that (1) is regular if there exists a complex pair $(z_1,z_2)$ such that $\det(z_1 z_2 E - z_1 A_1 - z_2 A_2)$ is not equal to zero, and simply regular if, for a complex $z$, either $\det(zE - A_1)$ or $\det(zE - A_2)$ is not equal to zero. It is easy to verify that simply regular implies regular, but regularity does not guarantee simple regularity. It is also known [8] that,

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there exists a unique solution to regular SMFM model if boundary conditions and inputs satisfy a certain set of boundary constraint relationship determined by the matrices $E, A_1, A_2, B_1$, and $B_2$.

Under the assumption of simple regularity of SMFM model, the generalized wave model has been derived [1]. In doing so, the non-singular equivalent of SMFM model was obtained via decomposition. Next, using the idea of wave model representation of 2-D state-space models [4,9], its equivalent 1-D format was established. At last, augmentation of equal size waves resulted in the generalized wave model (GWM).

The outline of the paper is as follows. In the following section, summary of procedures to come up with GWM is presented. In section 3, the general response formula for the model is derived in order to express the global state of the model purely in terms of inputs and initial boundary conditions within the admissible subspace. Section 4 is devoted to the concept of stability. Based on given definitions, sufficient conditions on global stability of the generalized wave model will be established. Finally, a numerical example is included to illustrate the effectiveness of GWM model and related theories through its MATLAB implementation.

## 2. The Generalized Wave Model Representation of The SMFM Model

From this point on, we are introducing the generalized wave model representation of (1) under the assumption of regularity of the pair $(E, A_1)$. It is worth mentioning that the same arguments hold true if $(E, A_2)$ is regular.

In [10], it was shown that for a pair of square size matrices $(E, A_1)$, if $\det(zE - A_1)$ is not equal to zero, or equivalently $(E, A_1)$ a regular pair, there exist nonsingular matrices $P$ and $Q$ such that:

$$PEQ = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N_{n_2} \end{bmatrix};$$

$$PA_1Q = \begin{bmatrix} c & 0 \\ 0 & I_{n_2} \end{bmatrix}; \quad n = n_1 + n_2. \tag{3}$$

where $n_1$ is the degree of the polynomial of $\det(zE - A_1)$, $I_x$ is the identity matrix of size $x$, and $N_{n_2}$ is the nilpotent matrix of size $n_2$. It is also important to define the index of a regular pair $(E, A_1)$, $\nu$, as the smallest integer such that:

$$N^{\nu-1} \neq 0, \text{and } N^{\nu} = 0 \tag{4}$$

In [1], it was shown that applying the same transformation to other matrices of the model resulted in definition of new variables

$$PA_2Q = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix};$$

$$PB_1 = \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix}; \; PB_2 = \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix}. \tag{5}$$

After simplification, the non-singular equivalent of SMFM model (1) was obtained as

$$\begin{aligned} x(i+1, j+1) &= \alpha_1 x(i+1, j) \\ &+ \sum_{l=0}^{\nu} \{\alpha_{l+2} x(i, j+l+1) \\ &+ \beta_{2l+1} u(i+1, j+l) \\ &+ \beta_{2l+2} u(i, j+l+1)\}. \end{aligned} \tag{6}$$

[Matrices are given in appendix (See A-1).] It was also shown that the set of inputs and boundary conditions (2) satisfying the following relation defines the admissible subspace.

$$\begin{aligned} &\begin{bmatrix} 0 & I_{n_2} \end{bmatrix} Q^{-1} x(i+1, 0) \\ &+ \sum_{l=1}^{\nu} N^{l-1} \{ \begin{bmatrix} d_{21} & d_{22} \end{bmatrix} Q^{-1} x(i, l) \\ &+ b_{21} u(i+1, l-1) + b_{22} u(i, l)\} = \underline{0}. \end{aligned} \tag{7}$$

**Theorem 1 [1]**

Under the regularity of the pair $(E, A_1)$, the singular system (1) has a unique solution, given by (6), if the set of boundary constraint relations (7) is satisfied for all $i \geq 0$.

In the wave model representation of 2-D state space models, states propagate in lines, increasingly in size, with slope of $-1$ in the first quadrant. In other words, being of first order, all local states $x(i,j)$ with $i + j - 1 = k$ lie on the $k$-*th* state of WAM. Indeed, knowledge of $k$ local states on $k$-th line suffices for computation of all $k+1$-*st* local states on the neighboring line $k+1$. However, from index dependent relation (6), it is obvious that by arranging $(\nu+1)$ equal size lines with the slope of $-1/(\nu+1)$, each wave front is of order $(\nu+1)$. So we start by defining new vectors for generally index $\nu$ system as follows

$t := \nu + 1 \; ; \; t' := 2\nu + 1$

$\chi^0(tk + l) := x(0, tk + l + 1)$

$\quad for \quad 0 \le l \le \nu \quad and \quad k \ge 0,$

$\chi_c^0(k) := x(k + 1, 0)$

$\quad for \quad k \ge 0 \,,$

$$\Phi(tk + l) := \begin{bmatrix} x(k+1, l) \\ x(k, t+l) \\ . \\ . \\ x(1, tk+l) \end{bmatrix}$$

$$for \quad 1 \le l \le t \quad and \quad k \ge 0 \,; \quad \in \Re^{n(k+1)}$$

$$U(tk + l) := \begin{cases} \begin{bmatrix} u(k, l+1) \\ u(k-1, t+l+1) \\ . \\ . \\ u(0, tk+l+1) \end{bmatrix} \\ for \quad 0 \le l \le \nu - 1 \quad \in \Re^{m(k+1)} \\ \qquad k \ge 0 \,. \\ \begin{bmatrix} u(k+1, 0) \\ u(k, t) \\ . \\ . \\ u(0, t(k+1)) \end{bmatrix} \\ for \quad l = \nu \quad \in \Re^{m(k+2)} \end{cases} \tag{8}$$

Rearranging the nonsingular equivalent of SMFM as given by (6), we have

$$\Phi(tk + l) = \begin{cases} \sum_{i=1}^{t} \{ M_i(k) \Phi(tk + 1 - i) \\ + H_i(k) \chi^0[t(k+1) - i] \} \\ + H_c(k) \chi_c^0(k) \\ + \sum_{i=1}^{t'} G_i(k) U[t(k+2) - 1 - i] \\ \quad for \quad l = 1, \quad k \ge 0 \,; \\ \sum_{i=1}^{t} \{ M_{[i+(l-1)t]}(k) \Phi(tk + l - i) \\ + H_i(k) \chi^0[t(k+1) + l - 1 - i] \} \\ + \sum_{i=1}^{t'} G_{[i+(l-1)t']}(k) U[t(k+1) + l + \nu - 1 - i] \\ \quad for \quad 2 \le l \le t, \quad k \ge 0. \end{cases}$$

$(See \; A-2 \; for \; newly \; defined \; matrices).$

(9)

Applying the same setting to (7), the boundary constraint relation becomes

$$\begin{aligned} &\delta \chi_c^0(i) + \sum_{l=1}^{\nu} \{ \omega_l \Phi(ti - t + l) \\ &\qquad + b_{21l}(i) U(ti + \nu + l - 1) \\ &\qquad + b_{22l}(i) U(ti + l - 1) \} = \underline{0} \;, \; \forall \; i \ge 0 \,. \end{aligned}$$

$(See \; A-3 \; for \; newly \; defined \; vectors \; and \; matrices).$

(10)

**Theorem 2 [1]**

Equation (9), completely characterizes (6) so it gives the recursive solution to SMFM (1) uniquely if boundary conditions (8) and all inputs involved, satisfy the set of boundary constraints (10) for all i >= 0.

Since the recursion formula in (9) is of order $(\nu + 1)$, we need to regroup family of equal size waves in a manner to end up with the generalized wave model. By augmenting waves of equal size together, the global state vector $X(k)$, becomes

$$X(k) := \begin{bmatrix} \Phi(tk-\nu) \\ \Phi(tk-\nu+1) \\ \cdot \\ \Phi(tk) \end{bmatrix} \quad for \quad k \geq 1 \qquad (11)$$

$$\in \Re^{tnk} \quad ; X(0) = \underline{0} \, ,$$

and rearranging the global boundary and input vectors as

$$X^0(k) := \begin{bmatrix} \chi_c^0(k) \\ \chi^0(tk) \\ \chi^0(tk+1) \\ \cdot \\ \chi^0(tk+2\nu) \end{bmatrix} \in \Re^{2tn} \; ;$$

$$V(k) := \begin{bmatrix} U(tk) \\ U(tk+1) \\ \cdot \\ \cdot \\ U(tk+3\nu) \end{bmatrix} \in \Re^{(3\nu+1)m(k+2)}$$

$$for \quad k \geq 0 \, , \qquad (12)$$

we have:

$$R_\nu(k+1)X(k+1) = L_\nu(k)X(k) + S_\nu(k)V(k) + J_\nu(k)X^0(k) \, , \qquad (13)$$

where $R_\nu(k)$ is square and invertible [See A-4 for newly defined matrices.]. Finally, if the pair $(E, A_1)$ is regular with index $\nu$, the generalized wave model representation of SMFM is of the form:

$$X(k+1) = F_\nu(k)X(k) + K_\nu(k)V(k) + C_\nu(k)X^0(k) \, . \qquad (14)$$

Using the above notations, the boundary constraint relation (10) is modified for the generalized wave model as

$$\Delta_\nu(i)X^0(i) + \Omega_\nu(i)X(i) + \Gamma_\nu(i)V(i) = \underline{0} \, , \; \forall \, i \geq 0 \, .$$
$$(See \; A-5 \; for \; newly \; defined \; matrices). \qquad (15)$$

**Theorem 3 [1]**

The generalized wave model (14) completely characterizes (9), so it gives the recursive solution to simply regular SMFM model (1) uniquely if boundary conditions $X^0(.)$ and all inputs involved $V(.)$ as defined in (11) satisfy the set of boundary constraint relations (15) for all $i \geq 0$.

## 3. The General Response Formula of the GWM

In order to obtain the general response formula for the generalized wave model, the state transition matrix, $T_\nu(l, j)$, is defined as follows

$$T_\nu(l,j) := \begin{cases} 0 & j = 0 \\ F_\nu(l)F_\nu(l-1)\ldots F_\nu(j) & 1 \leq j \leq l \\ I & j = l+1 \\ 0 & j \geq l+2. \end{cases} \qquad (16)$$

Starting with $k = 0$ in (14) we have:

$$X(1) = K_\nu(0)V(0) + C_\nu(0)X^0(0) \, . \qquad (17)$$

Repeating this procedure for increasing values of $k$, using (16) and substitution of (17) result in

$$X(2) = T_\nu(1,1)[K_\nu(0)V(0) + C_\nu(0)X^0(0)] + K_\nu(1)V(1) + C_\nu(1)X^0(1) \, , \qquad (18)$$

$$
\begin{aligned}
X(3) = {} & T_\nu(2,1)[K_\nu(0)V(0) \\
& + C_\nu(0)X^0(0)] + T_\nu(2,2)[K_\nu(1)V(1) \\
& + C_\nu(1)X^0(1)] \\
& + K_\nu(0)V(0) + C_\nu(0)X^0(0) ,
\end{aligned}
$$

$$\vdots \tag{19}$$

$$
\begin{aligned}
X(n) = {} & T_\nu(n-1,1)[K_\nu(0)V(0) \\
& + C_\nu(0)X^0(0)] \\
& + T_\nu(n-1,2)[K_\nu(1)V(1) + C_\nu(1)X^0(1)] \\
& + \ldots + T_\nu(n-1,j)[K_\nu(j-1)V(j-1) \\
& + C_\nu(j-1)X^0(j-1)] + \ldots \\
& + K_\nu(n-1)V(n-1) + C_\nu(n-1)X^0(n-1) .
\end{aligned} \tag{20}
$$

To simplify the notion, we express the general response formula as follows

$$
\begin{aligned}
X(n) = {} & \sum_{j=1}^{n} T_\nu(n-1,j)[K_\nu(j-1)V(j-1) \\
& + C_\nu(j-1)X^0(j-1)] \quad ; X(0) = \underline{0} .
\end{aligned} \tag{21}
$$

In a similar way, to formulate the set of boundary constraint relation (15) purely in terms of boundary conditions $X^0(.)$ and inputs $V(.)$, we have

$$\Delta_\nu(0)X^0(0) + \Gamma_\nu(0)V(0) = \underline{0} , \tag{22}$$

$$
\begin{aligned}
& [\Delta_\nu(1) \quad \Omega_\nu(1)C_\nu(0)] \begin{bmatrix} X^0(1) \\ X^0(0) \end{bmatrix} \\
& + [\Gamma_\nu(1) \quad \Omega_\nu(1)K_\nu(0)] \begin{bmatrix} V(1) \\ V(0) \end{bmatrix} = \underline{0} ,
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
& [\Delta_\nu(2) \quad \Omega_\nu(2)C_\nu(1) \quad \Omega_\nu(2)T_\nu(1,1)C_\nu(0)] \begin{bmatrix} X^0(2) \\ X^0(1) \\ X^0(0) \end{bmatrix} \\
& + [\Gamma_\nu(2) \quad \Omega_\nu(2)K_\nu(1) \quad \Omega_\nu(2)T_\nu(1,1)K_\nu(0)] \begin{bmatrix} V(2) \\ V(1) \\ V(0) \end{bmatrix} = \underline{0} ,
\end{aligned}
$$

$$\vdots \tag{24}$$

$$
\begin{aligned}
& [\Delta_\nu(i) \quad \Omega_\nu(i)C_\nu(i-1) \quad \Omega_\nu(i)T_\nu(i-1,i-1)C_\nu(i-2) \quad \ldots \cdot \ldots \quad \Omega_\nu(i)T_\nu(i-1,1)C_\nu(0)] \\
& \times \begin{bmatrix} X^0(i) \\ X^0(i-1) \\ X^0(i-2) \\ \vdots \\ X^0(0) \end{bmatrix} \\
& + [\Gamma_\nu(i) \quad \Omega_\nu(i)K_\nu(i-1) \quad \Omega_\nu(i)T_\nu(i-1,i-1)K_\nu(i-2) \quad \ldots \quad \Omega_\nu(i)T_\nu(i-1,1)K_\nu(0)] \\
& \times \begin{bmatrix} V(i) \\ V(i-1) \\ V(i-2) \\ \vdots \\ V(0) \end{bmatrix} = \underline{0} ,
\end{aligned} \tag{25}
$$

or equivalently

$$
\begin{aligned}
& \Delta_\nu(i)X^0(i) + \Gamma_\nu(i)V(i) \\
& + \Omega_\nu(i)\{ \sum_{j=0}^{i-1} T_\nu(i-1,j+1)[K_\nu(j)V(j) \\
& + C_\nu(j)X^0(j)]\} = \underline{0} \\
& ; \ \Omega_\nu(0) = 0 ; \quad \forall \ i \geq 0 ;
\end{aligned} \tag{26}
$$

## 4. Global Stability of the Generalized Wave Model

Although, the bounded-input-bounded-output (BIBO) stability of singular 2-D systems has been considered in [11], the issue of asymptotic stability, which is the most rigid type of stability is still untouched in the literature. In this regard, the generalized wave model is used to extend the results obtained on asymptotic stability of the wave model representation of standard 2-D systems to singular 2-D systems.

For this reason, we begin with the following definitions.

**Definition 1**

The regular system of equation (1) is said to be locally asymptotically stable if whenever $u(.) = 0$, there exists a finite pair of integers $N$ and $M$ such that

$$x(i,0) = 0 \quad \text{for } i \geq N \quad \text{and}$$

$$x(0,j) = 0 \quad \text{for } j \geq M$$

are from admissible subspace, and

$$\lim_{i \text{ and/or } j \to \infty} x(i,j) = 0.$$

**Definition 2**

The system of equation (14) is said to be globally asymptotically stable if whenever V(.)=0, there exists a finite integer L such that

$$X^0(i) = 0 \quad \text{for } i \geq L ,$$

is from admissible subspace, satisfying (15) and
$\lim_{i \to \infty} X(i) = 0$.

**Theorem 4**

Under the assumption of simple regularity of the pair $(E, A_1)$, the local asymptotic stability of SMFM model (1) is guaranteed by the global asymptotic stability of the generalized wave model (14).

Proof

Using theorems 1-3 and above definitions, it is obvious that as the global state of the GWM tends to zero, all local states of the original SMFM model would also converging to zero in the first quadrant. In fact, under the condition of regularity, the global stability of the GWM is sufficient for local stability of the SMFM model. **Q.E.D.**

Next, some sufficient condition for asymptotic stability of the GWM is established.

**Theorem 5**

The system of equation (14) is globally asymptotically stable if there exists a finite N such that

$$\|F_v(i)\| < 1 \qquad \forall i \geq N. \tag{27}$$

Proof

From definition 2, it is assumed that after a finite step, the input and boundary vectors vanish and

$$\|X(N+k+1)\| \leq$$
$$\|F_v(N+k) \cdot F_v(N+k-1) \cdot \ldots \cdot F_v(N)\| \cdot \|X(N)\|, \tag{28}$$
$$\leq \|F_v(N+k)\| \cdot \|F_v(N+k-1)\| \cdot \ldots \cdot \|F_v(N)\| \cdot \|X(N)\|, \tag{29}$$

*as* $k$ *goes to* $\infty$, $\|X(k)\|$ *approaches to zero, thus*

$$\lim_{k \to \infty} X(k) = 0. \qquad \text{Q.E.D.}$$

This theorem is equivalent to the so-called one-step test of stability as referred in [9]. On the other hand, the state transition matrix can be utilized to check the stronger condition, equivalent to m-step stability test [9].

**Theorem 6**

The system of equation (14) is globally asymptotically stable if for every $k$,

$$\lim_{k \to \infty} \|T_v(l,k)\| = 0. \tag{30}$$

Proof

Following the definition of the state transition matrix, $T_v(l,k)$, and according to equation (21), taking the limit from both sides, we obtain:

$$\lim_{n \to \infty} X(n) = \lim_{n \to \infty} \sum_{k=1}^{n} T_v(n-1,k)[K_v(k-1)V(k-1) + C_v(k-1)X^0(k-1)]. \tag{31}$$

Now, if the condition of the theorem is met for every $k$,

$$\lim_{n \to \infty} X(n) = 0,$$

and therefore the global state of the GWM converges to zero, hence the system is globally stable. **Q.E.D.**

Since the structure of $F_v(l)$ and therefore $T_v(l,k)$ matrices are index dependent and cannot be expressed uniquely, the following milder conditions may be checked instead.

**Corollary 1**

The system of equation (12) is globally asymptotically stable if

$$\lim_{k \to \infty} \sum_{i=1}^{l} \|M_{[i+(l-1)t]}(k)\| < 1 \qquad \forall 1 \leq l \leq t. \tag{32}$$

Proof

Using theorem 3 and equation (10.1),

$$\lim_{k \to \infty} X(k) = 0 \qquad \text{if} \qquad \lim_{k \to \infty} \Phi(k) = 0.$$

And from (8) and definition 2,

$$\lim_{k \to \infty} \|\Phi(tk+l)\| \leq \lim_{k \to \infty} \sum_{i=1}^{l} \|M_{i+(l-1)t}(k)\| \cdot \lim_{k \to \infty} \left\{ \max_i \|\Phi(tk+l-i)\| \right\} \qquad \forall\, 1 \leq l \leq t, \tag{33}$$

*Under this condition*

$$\lim_{k \to \infty} \|\Phi(tk+l)\| \leq \lim_{k \to \infty} \left\{ \max_i \|\Phi(tk+l-i)\| \right\} \qquad \forall\, 1 \leq l \leq t. \tag{34}$$

Hence, $\Phi(.)$ and therefore $X(.)$ are converging sequences. **Q.E.D.**

**Corollary 2**

The system of equation (12) is globally asymptotically stable if

$$\sum_{i=1}^{t+1} \|\alpha_i\| < 1. \tag{35}$$

Proof

By inspection, $\|M_{[(l-1)t+1]}(k)\| \le \|\alpha_l\| + \|\alpha_{l+1}\|$
for $1 \le l \le t$, (36)

and also $\|M_{[(l-1)t+i]}(k)\| \le \|\alpha_i\|$
for $1 \le l \le t$ and $2 \le i \le t$.(37)

Therefore

$$\sum_{i=1}^{t} \|M_{[(l-1)t+i]}(k)\| \le \sum_{i=1}^{t+1} \|\alpha_i\| \quad , for\ 1 \le l \le t \ . \quad (38)$$

Q.E.D.

## 5. Numerical Example

Consider a 2-D system modeled by(1) with the following matrices:

$$E = \begin{bmatrix} 25 & -5 & -14 & 16 \\ -3 & -5 & -10 & -6 \\ 12 & -4 & -8 & 6 \\ 28 & -8 & -20 & 16 \end{bmatrix};$$

$$A_1 = \begin{bmatrix} 6 & 14 & 22 & 18 \\ -3 & -5 & -8 & -7 \\ 1 & 5 & 10 & 5 \\ 6 & 14 & 22 & 18 \end{bmatrix};$$

$$A_2 = \begin{bmatrix} 2 & 4 & 6 & 5 \\ 1 & -3 & -4 & -2 \\ 1 & 0 & -1 & 1 \\ 3 & 3 & 4 & 5 \end{bmatrix};$$

$$B_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \\ -2 \end{bmatrix};$$

$$B_2 = \begin{bmatrix} 0 \\ -1 \\ -2 \\ -2 \end{bmatrix}. \quad (39)$$

We are interested to obtain the solution of local state $x(6,6)$ assuming that $u(1,1)=1$, and $x(0,1)=[1 \ \ 1 \ \ 1 \ \ 1]'$ are the only non-zero input and boundary condition on horizontal axis, respectively.

To check if the system is regular, we follow the decomposition algorithm to expand the polynomial of $\det(zE - A_1)$

$$|zE - A_1| = -24z^2 + 12z = -24(z^2 - 0.5z)$$
$$= -24z(z-0.5) \ne 0.(40) \quad (40)$$

Since the system is regular, we need to obtain other elements of decomposition as follows:

$$n_1 = 2\ ; n_2 = n - n_1 = 2 \quad ; c = \begin{bmatrix} 0 & 0 \\ 0 & 1/2 \end{bmatrix};$$

$$rank(E) = 3\ ; rank(N) = rank(E) - n_1 = 1 \quad ; \quad (41)$$

$$Eig(E) = \{37.15, -10.66, 0, 1.51\}$$

$;$ *Number of blocks of* $N$
$=$ *Number of zero Eigenvalues of* $E = 1$;

$$N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad \nu = 2;$$

$$P = \begin{bmatrix} -1 & 0 & 0 & 1 \\ -2 & -1 & 1/2 & 3/2 \\ 3 & 1 & 0 & -5/2 \\ -3 & -1 & -1 & 3 \end{bmatrix};$$

$$Q = \begin{bmatrix} -5/3 & 1 & -3 & 5/2 \\ -10/3 & 1 & -5 & 9/2 \\ 2/3 & 0 & 1 & -1 \\ 7/3 & -1 & 4 & -3 \end{bmatrix} \quad (42)$$

which would result in decomposed form as

$$\hat{E} = PEQ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix};$$

$$\hat{A}_1 = PA_1Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

$$\hat{A}_2 = PA_2Q = \begin{bmatrix} 1/3 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 1/4 \\ -1/3 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix} \quad (43)$$

$$\hat{B}_1 = PB_1 = [-1 \quad -1 \quad 3 \quad -6]' ;$$
$$\hat{B}_2 = PB_2 = [-2 \quad -3 \quad 4 \quad -3]'.$$

The non-singular equivalent of (1) would become as in (6) with the matrices

$$\alpha_1 = \begin{bmatrix} 1/2 & 1/2 & 2 & 1/2 \\ 1/2 & 1/2 & 2 & 1/2 \\ 0 & 0 & 0 & 0 \\ -1/2 & -1/2 & -2 & -1/2 \end{bmatrix};$$
$$\alpha_2 = \begin{bmatrix} -5/3 & 7/6 & 5/6 & 0 \\ -10/3 & 17/6 & 25/6 & 0 \\ 2/3 & -2/3 & -4/3 & 0 \\ 7/3 & -11/6 & -13/6 & 0 \end{bmatrix};$$
$$\alpha_3 = \begin{bmatrix} -4 & 9/2 & 29/2 & -1 \\ -7 & 15/2 & 49/2 & -2 \\ 3/2 & -3/2 & -5 & 1/2 \\ 5 & -6 & -19 & 1 \end{bmatrix}; \quad (44)$$

$$\alpha_4 = \begin{bmatrix} 3 & 0 & -3 & 3 \\ 5 & 0 & -5 & 5 \\ -1 & 0 & 1 & -1 \\ -4 & 0 & 4 & -4 \end{bmatrix};$$
$$\beta_1 = \begin{bmatrix} 2/3 \\ 7/3 \\ -2/3 \\ -4/3 \end{bmatrix}; \quad \beta_2 = \begin{bmatrix} 1/3 \\ 11/3 \\ -4/3 \\ -5/3 \end{bmatrix};$$
$$\beta_3 = \begin{bmatrix} 24 \\ 42 \\ -9 \\ -30 \end{bmatrix}; \quad \beta_4 = \begin{bmatrix} 39/2 \\ 67/2 \\ -7 \\ -25 \end{bmatrix};$$
$$\beta_5 = \begin{bmatrix} -18 \\ -30 \\ 6 \\ 24 \end{bmatrix}; \quad \beta_6 = \begin{bmatrix} -9 \\ -15 \\ 3 \\ 12 \end{bmatrix}.$$

Unique solution to (1) exists, if boundary conditions and inputs satisfy the following boundary constraint relations

$$\begin{bmatrix} 0 & 2 & 3 & 2 \\ 2 & 0 & -2 & 2 \end{bmatrix} x(i+1,0) + \begin{bmatrix} 3/2 & -1/2 & 0 & 1/2 \\ 1 & 0 & -1 & 1 \end{bmatrix} x(i,1) + \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x(i,2) + \begin{bmatrix} 3 & 4 & -6 & -3 \\ -6 & -3 & 0 & 0 \end{bmatrix} \begin{bmatrix} u(i+1,0) \\ u(i,1) \\ u(i+1,1) \\ u(i,2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\forall i \geq 0. \quad (45)$$

Since the desired local state $x(6,6)$ lies on $\Phi(21)$ or equivalently $X(7)$, using the general response formula of the GWM (21), we have

$$X(7) = \begin{bmatrix} \Phi(19) \\ \Phi(20) \\ \Phi(21) \end{bmatrix};$$
$$\Phi(21) = \begin{bmatrix} x(7,3) \\ x(6,6) \\ x(5,9) \\ x(4,12) \\ x(3,15) \\ x(2,18) \\ x(1,21) \end{bmatrix}; \quad x(6,6) = \begin{bmatrix} 0.1189 \\ 1.5347 \\ -0.7079 \\ -0.8268 \end{bmatrix}.$$

## 6. Conclusion

The general response formula for the generalized wave model has been derived. By doing so, the global state of the model is expressed purely as a function of inputs and boundary conditions from admissible subspace. Also, for the first time, the concept of asymptotic stability of singular 2-D systems was introduced and sufficient conditions for the global stability of the GWM were established.

The above considerations can be extended to study the controllability concept, the minimum energy optimal control problem, and to obtain stronger conditions on the stability of the model. Work is proceeding in these areas and are topics of future research.

## 7. APPENDIX

### A-1 Refer to Equation (6).

$$\alpha_1 := Q\begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix}Q^{-1}\ ; \quad \alpha_2 := Q\begin{bmatrix} d_{11} & d_{12} \\ 0 & 0 \end{bmatrix}Q^{-1}\ ; \quad \alpha_{l+2} := Q\begin{bmatrix} 0 & 0 \\ -N^{l-1}d_{21} & -N^{l-1}d_{22} \end{bmatrix}Q^{-1} \qquad 1 \le l \le v$$

$$\beta_1 := Q\begin{bmatrix} b_{11} \\ 0 \end{bmatrix}; \quad \beta_2 := Q\begin{bmatrix} b_{12} \\ 0 \end{bmatrix}; \quad \beta_{2l+1} := Q\begin{bmatrix} 0 \\ -N^{l-1}b_{21} \end{bmatrix}; \quad \beta_{2l+2} := Q\begin{bmatrix} 0 \\ -N^{l-1}b_{22} \end{bmatrix} \qquad 1 \le l \le v$$

$$M_{[n+l]}(k) := \begin{cases} \begin{bmatrix} \alpha_{v+2} & 0 & \cdot & \cdot & 0 \\ \alpha_1 & \alpha_{v+2} & 0 & \cdot & 0 \\ 0 & \alpha_1 & \alpha_{v+2} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \alpha_1 & \alpha_{v+2} \\ 0 & \cdot & \cdot & 0 & \alpha_1 \end{bmatrix} & for\ \ l=1,\ i=0,\ and\ k \ge 1 \quad \in \Re^{n(k+1)\times nk} \\ \begin{bmatrix} \alpha_1 & \alpha_{v+2} & 0 & \cdot & 0 \\ 0 & \alpha_1 & \alpha_{v+2} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & 0 & \alpha_1 & \alpha_{v+2} \\ 0 & \cdot & \cdot & 0 & \alpha_1 \end{bmatrix} & for\ \ l=1,\ 1 \le i \le v,\ and\ k \ge 0 \quad \in \Re^{n(k+1)\times n(k+1)} \\ \begin{bmatrix} \alpha_{v+3-l} & 0 & \cdot & \cdot & 0 \\ 0 & \alpha_{v+3-l} & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & \alpha_{v+3-l} \\ 0 & \cdot & \cdot & 0 & 0 \end{bmatrix} & for\ \ 2 \le l \le t,\ 0 \le i \le l-1,\ and\ k \ge 1 \quad \in \Re^{n(k+1)\times nk} \end{cases}$$

### A-2 Refer to Equation (9).

$$M_{[n+l]}(k) = \left\{ \begin{bmatrix} 0 & \alpha_{v+3-l} & 0 & \cdot & 0 \\ 0 & 0 & \alpha_{v+3-l} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & \alpha_{v+3-l} \\ 0 & \cdot & \cdot & 0 & 0 \end{bmatrix} \right. \quad for\ 2 \le l \le v,\ l \le i \le v,\ and\ k \ge 0 \quad \in \Re^{n(k+1)\times n(k+1)}$$

$$G_{[t'i+l]}(k) :=$$

$$
\begin{cases}
\begin{bmatrix} \beta_{2\nu+3-2l} & 0 & 0 & \cdot & 0 \\ 0 & \beta_{2\nu+3-2l} & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & 0 & \beta_{2\nu+3-2l} & 0 \end{bmatrix} & \text{for } 1 \le l \le \nu, \text{ and } 0 \le i \le l-1 \quad \in \Re^{n(k+1) \times m(k+2)} \\
\begin{bmatrix} 0 & \beta_{2\nu+3-2l} & 0 & \cdot & 0 \\ 0 & 0 & \beta_{2\nu+3-2l} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & 0 & \beta_{2\nu+3-2l} & 0 \end{bmatrix} & \text{for } 1 \le l \le \nu, \text{ and } l \le i \le \nu \quad \in \Re^{n(k+1) \times m(k+3)} \\
\begin{bmatrix} \beta_1 & \beta_{2\nu+2} & 0 & \cdot & 0 \\ 0 & \beta_1 & \beta_{2\nu+2} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & 0 & \beta_1 & \beta_{2\nu-2} \end{bmatrix} & \text{for } l = t, \text{ and } 0 \le i \le \nu \quad \in \Re^{n(k+1) \times m(k+2)} \\
\begin{bmatrix} \beta_{2(2t-l)} & 0 & 0 & \cdot & 0 \\ 0 & \beta_{2(2t-l)} & 0 & \cdot & 0 \\ 0 & 0 & \beta_{2(2t-l)} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 0 & \beta_{2(2t-l)} \end{bmatrix} & \text{for } t+1 \le l \le t', \text{ and } 0 \le i \le l-t-1 \quad \in \Re^{n(k+1) \times m(k+1)} \\
\begin{bmatrix} 0 & \beta_{2(2t-l)} & 0 & \cdot & 0 \\ 0 & 0 & \beta_{2(2t-l)} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & \beta_{2(2t-l)} \end{bmatrix} & \text{for } t+1 \le l \le t', \text{ and } l-t \le i \le \nu \quad \in \Re^{n(k+1) \times m(k+2)}
\end{cases}
$$

$$H_i(k) := \begin{bmatrix} 0 \\ \cdot \\ 0 \\ \alpha_{\nu+3-i} \end{bmatrix} \qquad \text{for } 1 \le i \le t \text{ ; and} \qquad H_c(k) := \begin{bmatrix} \alpha_1 \\ 0 \\ \cdot \\ 0 \end{bmatrix} \qquad k \ge 0 \qquad \in \Re^{n(k+1) \times n} \quad \textbf{A-}$$

**A-3 Refer to Equation (10)**

$$\delta := \begin{bmatrix} 0 & I_{n_2} \end{bmatrix} Q^{-1} \qquad \in \Re^{n_2 \times n}$$

$$\omega_l := N^{l-1} \begin{bmatrix} d_{21} & d_{22} \end{bmatrix} Q^{-1} \qquad \in \Re^{n_2 \times n} \qquad 1 \le l \le \nu$$

$$b_{21l}(i) := \begin{bmatrix} N^{l-1} b_{21} & 0 & \dots & 0 \end{bmatrix} \qquad \in \Re^{n_2 \times (i+2)m} \qquad 1 \le l \le \nu$$

$$b_{22l}(i) := \begin{bmatrix} N^{l-1} b_{22} & 0 & \dots & 0 \end{bmatrix} \qquad \in \Re^{n_2 \times (i+1)m} \qquad 1 \le l \le \nu$$

$$\Phi(-l) := \chi^0(\nu - l) \qquad 1 \le l \le \nu$$

## A-4 Refer to Equation (13)

$$R_\nu(k+1) := \begin{bmatrix} I & 0 & 0 & \dots & 0 \\ -M_{[t^2-\nu]}(k) & I & 0 & \dots & 0 \\ . & . & \dots & . & . \\ -M_{[t^2-2]}(k) & \dots & -M_{[t^2-\nu]}(k) & I & 0 \\ -M_{[t^2-1]}(k) & \dots & -M_{[t^2+1-\nu]}(k) & -M_{[t^2-\nu]}(k) & I \end{bmatrix} \in \Re^{tn(k+1)\times tn(k+1)}$$

$$L_\nu(k) := \begin{bmatrix} M_t(k) & M_{t-1}(k) & \dots & \dots & M_1(k) \\ 0 & M_t(k) & \dots & \dots & M_2(k) \\ . & . & . & . & . \\ 0 & \dots & 0 & M_t(k) & M_{t-1}(k) \\ 0 & \dots & \dots & 0 & M_t(k) \end{bmatrix} \in \Re^{tn(k+1)\times tnk}$$

$$S_\nu(k) := \begin{bmatrix} G_{t'}(k) & G_{t'-1}(k) & \dots & G_2(k) & G_1(k) & 0 & \cdot\ \cdot & 0 \\ 0 & G_{2t'}(k) & G_{2t'-1}(k) & \dots & G_{t'+2}(k) & G_{t'+1}(k) & 0\ \cdot & 0 \\ . & . & . & . & . & . & .\ . & . \\ 0 & . & . & 0 & G_{tt'}(k) & G_{tt'-1}(k) & \cdot\ \cdot & G_{\nu t'+1}(k) \end{bmatrix} \in \Re^{tn(k+1)\times(3\nu+1)m(k+2)}$$

$$J_\nu(k) := \begin{bmatrix} H_c(k) & H_t(k) & H_t(k) & \dots & \dots & H_t(k) & H_t(k) & 0 & \dots & 0 \\ 0 & 0 & H_t(k) & H_t(k) & \dots & \dots & H_t(k) & H_t(k) & . & 0 \\ . & . & . & . & . & . & . & . & . & . \\ 0 & \dots & \dots & \dots & \dots & 0 & H_t(k) & \dots & H_t(k) & H_t(k) \end{bmatrix} \in \Re^{tn(k+1)\times 2tn}$$

## A-5 Refer to Equation (15)

$$F_\nu(k) := R_\nu^{-1}(k+1)L_\nu(k);\ K_\nu(k) := R_\nu^{-1}(k+1)S_\nu(k);\ C_\nu(k) := R_\nu^{-1}(k+1)J_\nu(k);$$

$$\Omega_\nu(i) := \begin{cases} 0 & \text{for } i=0 \\ [\omega_1\ \ \omega_2\ \ \dots\ \ \omega_\nu\ \ 0] & \text{for } i\ge 1 \end{cases} \in \Re^{n_2\times tni}$$

$$\Delta_\nu(i) := \begin{cases} [\delta\ \ \omega_1\ \ \omega_2\ \ \dots\ \ \omega_\nu\ \ 0\ \ \dots\ \ 0] & \text{for } i=0 \\ [\delta\ \ 0\ \ \dots\ \ 0] & \text{for } i\ge 1 \end{cases} \in \Re^{n_2\times 2tn}$$

$$\Gamma_\nu(i) := [b_{221}(i)\ \dots\ b_{22\nu}(i)\ \ b_{211}(i)\ \dots\ b_{21\nu}(i)\ \ 0\ \dots\ 0] \in \Re^{n_2\times(3\nu+1)m(i+2)}$$

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