HKODelay-dependent Stability Analysis of Nonlinear Time-invariant Time-Delay Systems

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ABSTRACT

This paper presents delay-dependent stability analysis for some types of nonlinear time-delay systems. Motivated by Lyapunov-Krasovskii functionals and the related theorems on stability of time-delay systems, we introduce a new procedure for generating such functionals. The main reason for considering Lyapunov-Krasovskii functionals is the less conservatism of this method. The obtained conditions are sufficient and local. Applying the introduced procedure on a typical system with time-invariant parameters derives more results and a numerical example presents the capability of the method.

KEYWORDS

Nonlinear Time-Delay, Delay-dependent stability, Lyapunov Functional

1. INTRODUCTION

Many control systems in various fields such as aerospace, biology, mechanics, economic, chemical as well as process control systems often experience a phenomenon called "time-delay". Delay may appear either in the state variables, the control input, or the system output. This phenomenon changes the dynamic equations governing such systems from those of a nonlinear one (i.e. the ordinary differential equations). The time-delay, in many cases, acts as a source of instability. Unlike ordinary differential equations, time-delay dynamic equations are infinite dimensional. Therefore from both theoretical and practical aspects, the performance of time-delay control systems and stability issue are of great importance.

The time delay equations studies first began in the 18th century. Some geometrical models were introduced by Euler, Condorect and Bernoulli. V.Volterra (1920) and others like R.Bellman and A.Myshikis followed the work in a more advanced way till 1950 [1]. At the end of 1950, N.Krasovskii and B. Razumikhin opened a new chapter of studies about the stability of time-delay systems by introducing Lyapunov functionals and Lyapunov functions respectively [1,2]. It was the point after which the discussions became much more complex in related fields such as mathematics, biology and control engineering.

The stability region of the time-delay systems is divided into two areas. Within one area, the system is stable regardless to the amount of delay (delay-independent stability regain) while within the second area, delay has significant effects on stability (delay-dependent stability regain). Delay-dependent stability has a less conservative constraints than delay-independent stability, so due to this fact, the controllers of such systems which are designed according to the results of delay-independent stability analysis need an over design. For the systems with unknown delay, using the delay-dependent stability conditions is a more suitable choice [1,3].

It is very difficult to make a good choice of Lyapunov-Krasovskii functionals to obtain the stability conditions. In another word, in addition to the complexity of analytical discussion of such systems with respect to the ordinary systems, the calculation of Lyapunov functionals is more difficult than that of Lyapunov functions. The general form of Lyapunov-Krasovskii functionals leads to a complicated system of partial differential equations [4]. The special forms of Lyapunov-Krasovskii functionals lead to simpler delay-independent [5,6,7] and delay-dependent conditions [8,9,10].

Most of the researches accomplished so far (as the least knowledge of the authors) are related to the stability of linear time-delay systems. The most used method in stability analysis of the linear systems, in frequency domain is root locus [11] and in time domain is Lyapunov-Krasovskii functionals [2,12]. The simple forms of Lyapunov-Krasovskii functionals introduce the delay-

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independent stability, which is sufficient for analysis of systems, where the coefficient of the delay term is small. If the delay is small, the system with discrete delay is converted to a system with distributed delay by model transformation methods. Then the delay-dependent stability condition of the main system is determined by applying Razumikhin stability criterion or Lyapunov-Krasovskii functionals on the transformed system [13]. If the delay is not small, the method introduces a great conservatism. The main reason of this conservatism is that the transformed model may make additional poles one of which possibly cut the imaginary axis before the main system poles [14,15]. Another reason of conservatism is to make additional assumptions to obtain the stability conditions of the transformed model [8]. In recent years, in order to investigate the delay-dependent stability of such systems, a new transformation method has been introduced called "*descriptor model*" [3]. In contrast to preceding transformations, the descriptor model leads to a system, which is equivalent to the main system and is not dependent on additional assumptions for stability. It should be noted that the most of the present researches on stability analysis of linear time-delay systems are related to analysis and synthesis of uncertainty systems with time delay [10]. This uncertainty may appear either in system parameters or delay. The major method for the robust stability analysis of such systems is Lyapunov-Krasovskii functionals. In recent researches, efforts have been done to reduce the conservatism of the robust stability results by modification of these functionals [10,13,15]. The results obtained by these functionals are considered as sufficient condition for robust stability. If there is no uncertainty in the systems, these results would be both the necessary and sufficient conditions [15,16,17].

Carried out researches on stability analysis of nonlinear time-delay systems has not developed as linear time-delay systems. The results obtained for the stability of such systems are more about the delay-independent stability where many assumptions should be made to receive the results [18]. The reference [19] discusses robust stabilization of a class of nonlinear time-delay systems in order to design a stabilizer controller using Lyapunov-Krasovskii functional. It showed that they have some mistakes in their derivation [20]. As far as the author knows, there are a few literatures in the field of delay-dependent stability analysis. For example, there is a paper in this field that discuss delay-dependent control for time-delayed T-S Fuzzy systems using descriptor representation [21].

The present paper introduces an algorithm for constructing Lyapunov-Krasovskii functionals by which it could be possible to analyze locally the delay-dependent stability of some nonlinear time-delay systems. The main method used here is to expand the concept of Lyapunov-Krasovskii functionals. The final results have been obtained by applying the linear matrix inequalities (LMI). The method has been also applied to a class of nonlinear time-delay systems in which there is a stable linear term in scalar case [24,25]. The main advantage of this method is that it could be possible to expand it for other types of nonlinear time-delay systems. So, the applied assumptions on the system is limited and finally, it is possible which we derived delay-independent stability results for some cases.

Section 2 describes the time-delay systems and their stability theorem. The next section explains the main problem and how to construct a proper Lyapunov functional. The final results are applied to the scalar system with constant parameters in section 4. Section 5 includes a numerical example. Finally, the conclusions and some propositions are offered in section 6.

2. Time-Delay Systems and Their Stability [2]

Generally, time-delay systems are represented as follows :

$$\begin{cases} \dot{x}(t) = f(t, x_t(\theta)) & , \quad t \geq 0 \\ x_t(\theta) = x(t+\theta) & , \quad -h \leq \theta \leq 0 \end{cases} \tag{1}$$

where $x \in \Re^n$ is the state vector of system, $h \in \Re^+$ is the delay, f is a function of class C, $x(t) = \varphi(t)$ is the initial condition of the system for the time interval $-h \leq \theta \leq 0$ and x_t is of class C for $t \geq 0$. Having an appropriate Lyapunov functional $V : \Re_+ \times C \to \Re$, the sufficient condition for stability of (2-1) will be [2]:

$$i) \quad u(|\varphi(0)|) \leq V(t, \varphi) \tag{2}$$

$$ii) \quad \dot{V}(t, \varphi) \leq 0 \tag{3}$$

where $|\varphi(0)| = \max\limits_{-h \leq \tau \leq 0} |\varphi(\tau)|$ and $u : \Re^+ \to \Re^+$ is a function for which $u(0) = 0$ and $u(s) > 0 \ \ (s \in \Re^+)$.

In order to establish the stability condition, the offered Lyapunov functional should meet conditions (2) and (3) for $t \geq 0$.

3. The Problem Statement

The time-delay nonlinear dynamic equation is defined as follows:

$$\dot{x}(t) = BF(x(t)) + CF(x(t-h)) \ , \ t \geq 0 \tag{4}$$

where $x \in \Re^n$ is the state vector, $h \in \Re^+$ is the delay and B and C are the diagonal matrices with real elements and have the relation $B + C < 0$. $x = 0$ is the equilibrium state of the system (4). $F = [f_1 \ \ f_2 \ \ \ldots \ \ f_n]^T$ is a vector function with n components which are continuous functions of the state vector. It is assumed for this system that the following relation is true for each component of the vector $y \in \Re^n - \{0\}$:

$$0 < \frac{f_i(y)}{y_i} \leq m_i \ \ , \ \ that \ \ i = 1, 2, \ldots, n \tag{5}$$

where each $m_i (i=1,2,...,n)$ is positive constant. It should be noted that due to the existing of condition (5), the system stability would be treated locally. Note that the condition (5) is a special case of Lipschitz condition. From the relation (5), one has:

$$|f_i(y)| \le m_i |y_i| \quad , \quad that \quad y_i \ne 0 \quad , \quad i=1,2,...,n$$

Regarding this fact that y and F are n vector components, we can prove the following relations for any given negative definite diagonal matrix $K \in \Re^{n \times n}$:

$$y^T KF \le F^T M^{-1} KF \quad or \quad F^T Ky \le F^T KM^{-1} F \qquad (6)$$

where $M = diag(m_i)$ is a positive diagonal matrix [26].

The Lyapunov functional construction algorithm:

The Lyapunov functional $V: C \to \Re$ is assumed as follows:

$$V(x_t) = V_1(x_t) + V_2(x_t)$$

where

$$V_1 = \frac{1}{2}\left[x(t) + \int_{t-h}^{t} R(t,u)F(x(u))du\right]^T \left[x(t) + \int_{t-h}^{t} R(t,u)F(x(u))du\right]$$

$R(t,u)$ is an unknown non-constant $n \times n$ matrix with time-dependent coefficients which is determined as the algorithm is proceeded. If this matrix is positive definite, then V_1 will also be positive definite and meet the required constraint (2). On the other hand, if V_1 can be a Lyapunov functional (i.e. conditions (2) and (3) are hold) we can just derive the delay-independent stability condition. In other word, because we apply no limits on the matrix $R(t,u)$, it might be possible that V_1 does not satisfy constraint (2). To solve this problem there should some assumption be made on $R(t,u)$, so that V_1 become positive definite. Also we can select V_2 in a way that Lyapunov functional V satisfies constraint (2). Now, we should first calculate the derivative of V_1 before to reach the final conclusion. So, we will have:

$$\dot{V}_1 = \frac{1}{2}\left[x(t) + \int_{t-h}^{t} R(t,u)F(x(u))du\right]^T \times \left[\int_{t-h}^{t} \frac{\partial R(t,u)}{\partial t} F(x(u))du + BF(x(t)) + R(t,t)F(x(t)) + (C - R(t,t-h))F(x(t-h))\right]$$

$$+ \frac{1}{2}\left[\int_{t-h}^{t} \frac{\partial R(t,u)}{\partial t} F(x(u))du + BF(x(t)) + R(t,t)F(x(t)) + (C - R(t,t-h))F(x(t-h))\right]^T \times \left[x(t) + \int_{t-h}^{t} R(t,u)F(x(u))du\right]$$

for simplicity, we assume :

$$R(t,t-h) = C \qquad (7)$$

If we consider the scalar case for the problem ($x \in \Re$), then $R(t,u)$ would be a three-dimensional surface with independent variables t and u. For each $t \in \Re^+$, this surface would be of distance C from $t-u$ plane at the points $(t,u) = (t, t-h)$.

In order to put $\dot{V}_1$ in quadratic form, the first step is to let $\frac{\partial R}{\partial t}$ in terms of R. Therefore, $\frac{\partial R}{\partial t}$ is considered as follows:

$$\frac{\partial R(t,u)}{\partial t} = \Psi(t)R(t,u)$$

$\Psi(t)$ is an unknown matrix and for simplicity could be assumed as a constant matrix Q:

$$\frac{\partial R(t,u)}{\partial t} = QR(t,u)$$

Applying the initial condition (7), matrix function R would be as follows:

$$R(t,u) = Ce^{Q(u-t+h)}$$

Q is an unknown real matrix which it's structure is determined in the procedure of obtaining the stability of the system (4). We define the following two relationships to simplify the relations:

$$\begin{cases} F_1(x(t)) = \int_{t-h}^{t} R(t,u)F(x(u))du = C\int_{t-h}^{t} e^{Q(u-t+h)}F(x(u))du \\ B_1 = B + Ce^{Qh} \end{cases}$$

Then $\dot{V}_1$ would be as follows:

$$\dot{V}_1 = x^T(t)B_1F(x(t)) - F_1^T(x(t))Q^T x(t) + F_1^T(x(t))B_1F(x(t)) - F_1^T(x(t))QF_1(x(t))$$

Remark (1): Q is the only arbitrary matrix in V_1 and $\dot{V}_1$ which can be determined. Because, the sign of $\dot{V}_1$ is not important, the structure of Q is determined in such away that V_1 meet the constraint (2). Therefore, if the constraint (2) is satisfied by V, it must be also satisfied by V_2 in addition to V_1. So, the optimum choice of V_2 can be made according to the following lemma:

Lemma (1) [22] *:* For the continuous function $g(t)$, one has the following equation:

$$\frac{d}{dt}\left[\int_{-h}^{0}\int_{t+s}^{t} g(u)duds\right] = hg(t) - \int_{-h}^{0} g(t+s)ds = hg(t) - \int_{t-h}^{t} g(\tau)d\tau$$

in which h is some positive constant.

The following relation for V_2 could be considered according to above lemma:

$$V_2 = \int_{-h}^{0}\int_{t+s}^{t} F^T(x(u))Pe^{Qs}F(x(u))duds$$

where P is a constant unknown $n\times n$ matrix which is assumed to be positive definite. Thus, the derivative of V_2 would be as follows:

$$\begin{aligned}\dot{V}_2 &= \int_{-h}^{0}\left[F^T(x(t))Pe^{Qs}F(x(t))\right]ds \\ &\quad - \int_{-h}^{0}\left[F^T(x(t+s))Pe^{Qs}F(x(t+s))\right]ds \\ &= F^T(x(t))PQ^{-1}\left[I - e^{-Qh}\right]F(x(t)) \\ &\quad - \int_{t-h}^{t} F^T(x(u))Pe^{Q(u-t)}F(x(u))du\end{aligned} \tag{8}$$

Now, the last term of the above relation should be converted to a quadratic form of F_1.

Lemma (2) [23]***: Linear Matrix Inequalities (LMI)***

Schur complement : Let $G: V \rightarrow S$ is partitioned according to:

$$G(x) = \begin{bmatrix} G_{11}(x) & G_{12}(x) \\ G_{21}(x) & G_{22}(x) \end{bmatrix}.$$

V is the vector space and S is the set of matrices which: $S = \{M \mid \exists n > 0, M = M^T \in \Re^{n\times n}\}$

It is also assumed that $G_{11}(x)$ is a $r\times r$ non-singular matrix. So, the matrix $T := G_{22} - G_{21}G_{11}^{-1}G_{12}$ is called the Schur complement of G_{11} in G. Then $G(x) > 0$ if and only if

$$\begin{cases} G_{11}(x) > 0 \\ G_{22}(x) - G_{12}(x)[G_{11}(x)]^{-1}G_{21}(x) > 0 \end{cases} \tag{9}$$

Note that the second inequality in (9) is a non-linear matrix inequality in x.

Applying the lemma (2) into the second term of the equation (8), one make the following matrix :

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \tag{10}$$

Where

$$A_1 = \left(\int_{t-h}^{t} e^{Qu}du\right)P^{-1}e^{Q^T t}, \quad A_2 = \int_{t-h}^{t} F^T(x(u))e^{Q^T u}du$$

$$A_3 = \int_{t-h}^{t} e^{Qu}F(x(u))du$$

$$A_4 = \int_{t-h}^{t} F^T(x(u))Pe^{Q(u-t)}F(x(u))du$$

If Q and P would be diagonal with nonzero elements, then the matrix A will be symmetrical and so the conditions of the lemma (2) satisfied. We can show that the matrix A is positive definite [15]. Thus, the relation (9) is satisfied by the matrix A. In other words, by applying the lemma (2) on the matrix A, one has :

$$\begin{aligned}&\int_{t-h}^{t} F^T(x(u))Pe^{Q(u-t)}F(x(u))du \\ &\geq \left(\int_{t-h}^{t} F^T(x(u))e^{Qu}du\right) \left(\left[\int_{t-h}^{t} e^{Qu}du\right]P^{-1}e^{Qt}\right)^{-1}\left(\int_{t-h}^{t} e^{Qu}F(x(u))du\right) \\ &\geq \left(\int_{t-h}^{t} F^T(x(u))e^{Qu}du\right)\left(e^{-Qt}P\left[I - e^{-Qh}\right]^{-1}Qe^{-Qt}\right) \left(\int_{t-h}^{t} e^{Qu}F(x(u))du\right)\end{aligned}$$

the right hand side of the inequality is set as follows:

$$\left(\int_{t-h}^{t} F^T(x(u))e^{Q(u-t+h)}Cdu\right) \left(C^{-1}e^{-Qh}P\left[I - e^{-Qh}\right]^{-1}Qe^{-Qh}C^{-1}\right) \left(\int_{t-h}^{t} Ce^{Q(u-t+h)}F(x(u))du\right)$$

then, one has:

$$\int_{t-h}^{t} F^T(x(u))Pe^{Q(u-t)}F(x(u))du$$
$$\geq F_1^T(x(t))\left[C^{-1}e^{-Qh}P\left[I-e^{-Qh}\right]^{-1}Qe^{-Qh}C^{-1}\right] F_1(x(t))$$

thus, one can write:

$$\dot{V}_2 \leq F^T(x(t))PQ^{-1}\left[I-e^{-Qh}\right]F(x(t)) - F_1^T(x(t))\left[C^{-1}e^{-Qh}P\left[I-e^{-Qh}\right]^{-1}Qe^{-Qh}C^{-1}\right] F_1(x(t)$$

Finally, the derivative of the Lyapunov functional V will be obtain as follows:

$$\begin{aligned}\dot{V} \leq\ & x^T(t)B_1F(x(t)) - F_1^T(x(t))Qx(t) \\ & + F_1^T(x(t))B_1F(x(t)) \\ & + F^T(x(t))PQ^{-1}\left[I-e^{-Qh}\right]F(x(t)) - \\ & - F_1^T(x(t))\left[Q + C^{-1}e^{-Qh}P\left[I-e^{-Qh}\right]^{-1}Qe^{-Qh}C^{-1}\right] \\ & F_1(x(t))\end{aligned} \tag{11}$$

If the diagonal matrices B_1 and $-CQ$ are negative definite, then by using the relation (6), the following inequalities are hold :

$$x^T(t)B_1F(x(t)) \leq F^T(x(t))M^{-1}B_1F(x(t)) \tag{12}$$
$$-F_1^T(x(t))Qx(t) \leq -F_1^T(x(t))QM^{-1}F(x(t)) \tag{13}$$

Applying (12) and (13) into (11), one has:

$$\begin{aligned}\dot{V} \leq\ & F^T(x(t))\left[M^{-1}B_1 + PQ^{-1}\left[I-e^{-Qh}\right]\right]F(x(t)) \\ & + F_1^T(x(t))\left[B_1 - QM^{-1}\right]F(x(t)) \\ & - F_1^T(x(t))\left[Q + C^{-1}e^{-Qh}P\left[I-e^{-Qh}\right]^{-1}Qe^{-Qh}C^{-1}\right] \\ & F_1(x(t)\end{aligned}$$

In order to be converted $\dot{V}$ to the quadratic form in terms of F or F_1, one should have the following inequalities [26]:

$$\begin{aligned} &i)\quad Q + C^{-1}e^{-Qh}P\left[I-e^{-Qh}\right]^{-1}Qe^{-Qh}C^{-1} \geq 0 \\ &ii)\quad M^{-1}B_1 + PQ^{-1}\left[I-e^{-Qh}\right] \leq 0\end{aligned} \tag{14}$$

If the inequality (i) satisfied, then the following inequality can be proved using the LMI algorithm [26] :

$$\begin{aligned} & -F_1^T(x(t))\left[Q + C^{-1}e^{-Qh}P\left[I-e^{-Qh}\right]^{-1}Qe^{-Qh}C^{-1}\right]F_1(x(t)) \\ & \quad + F_1^T(x(t))\left[B_1 - QM^{-1}\right]F(x(t)) \\ & \leq F^T(x(t))\left[\frac{1}{4}\left(B_1 - M^{-1}Q\right)\right. \\ & \quad \times\left(Q + C^{-1}e^{-Qh}P\left[I-e^{-Qh}\right]^{-1}Qe^{-Qh}C^{-1}\right)^{-1} \\ & \quad \left.\times\left(B_1 - QM^{-1}\right)\right]F(x(t))\end{aligned} \tag{15}$$

Finally, $\dot{V}$ is expressed as follows:

$$\begin{aligned}\dot{V} \leq\ & F^T(x(t))\left[M^{-1}B_1 + PQ^{-1}\left[I-e^{-Qh}\right]\right. \\ & + \frac{1}{4}\left(B_1 - M^{-1}Q\right) \\ & \left(Q + C^{-1}e^{-Qh}P\left[I-e^{-Qh}\right]^{-1}Qe^{-Qh}C^{-1}\right)^{-1} \\ & \left.\left(B_1 - QM^{-1}\right)\right]F(x(t))\end{aligned} \tag{16}$$

or, on the other hand :

$$\begin{aligned}\dot{V} \leq\ & \lambda_M\left[M^{-1}B_1 + PQ^{-1}\left[I-e^{-Qh}\right]\right. \\ & + \frac{1}{4}\left(B_1 - M^{-1}Q\right) \\ & \times\left(Q + C^{-1}e^{-Qh}P\left[I-e^{-Qh}\right]^{-1}Qe^{-Qh}C^{-1}\right)^{-1} \\ & \left.\times\left(B_1 - QM^{-1}\right)\right]\left\|F(x(t))\right\|_2^2\end{aligned} \tag{17}$$

where $\lambda_M[D]$ is the maximum eigenvalue of the square matrix D.

Remark (2) : In order to reach (16) (or (17)) and also obtain the minimum necessary conditions for $\dot{V}$ to be negative definite, it is necessary that (14) is satisfied. The limits on matrix P can be determined using the relations (14). Undoubtedly, in order to find the exact limits of P and Q, we need to analyze the relations (14) more deeply. We do this investigation for the scalar case in the next section.

Choosing P and Q from relations (14) properly, the system (4) is Lyapunov stable if:

$$\begin{aligned} & \lambda_M\left[M^{-1}B_1 + PQ^{-1}(I-e^{-Qh})\right. \\ & + \frac{1}{4}\left(B_1 - M^{-1}Q\right) \\ & \times\left(Q + C^{-1}e^{-Qh}P\left[I-e^{-Qh}\right]^{-1}Qe^{-Qh}C^{-1}\right)^{-1} \\ & \left.\times\left(B_1 - QM^{-1}\right)\right] \leq 0\end{aligned} \tag{18}$$

Using the above relation, we can obtain the boundary of matrix M. (regain of attraction) for the local stability of the system (4). The simulation results showed that there is at least a proper P and Q for the stable systems for which (18) is satisfied.

4. THE SCALAR CASE OF THE PROBLEM

The scalar time-delay nonlinear dynamic equation (4) is considered as follows:

$$\dot{x}(t) = bf(x(t)) + cf(x(t-h)) \quad , \quad t \geq 0 \qquad (19)$$

where $x \in \Re$ is the state variable, $h \in \Re^+$ is the delay and b, c are the real parameters of the system (19) that $b + c < 0$. $x = 0$ is the equilibrium point of (17), f is a continuous function and the relation $0 \leq \frac{f(y)}{y} \leq m$ (for each $y \in \Re - \{0\}$) is satisfied for the system (19) (Lipschitz condition).

V_1 and V_2 are considered as follows:

$$V_1 = \frac{1}{2}\left[x(t) + \int_{t-h}^{t} r(t,u) f(x(u)) du\right]^2$$

$$V_2 = \int_{-h}^{0} \int_{t+s}^{t} pe^{qs} f^2(x(u)) du ds$$

Applying the procedure of preceding section, one has:

$$\dot{V}_1 = b_1 x(t) f(x(t)) - qx(t) f_1(x(t)) + b_1 f(x(t)) f_1(x(t)) - q f_1^2(x(t))$$

$$\dot{V}_2 = pf^2(x(t)) \int_{t-h}^{t} e^{qs} ds - pe^{-qt} \int_{t-h}^{t} e^{qu} f^2(x(u)) du$$

Using Schwarz integral inequality (instead of LMI algorithm), we have:

$$\left[\int_{t-h}^{t} e^{qu} f(x(u)) du\right]^2 \leq \int_{t-h}^{t} e^{qu} du \times \int_{t-h}^{t} e^{qu} f^2(x(u)) du$$

thus:

$$\int_{t-h}^{t} e^{qu} f^2(x(u)) du \geq \frac{\left[\int_{t-h}^{t} e^{qu} f(x(u)) du\right]^2}{\int_{t-h}^{t} e^{qu} du} = \frac{e^{qt-2qh}}{c^2 \int_{-h}^{0} e^{qs} ds} f_1^2(x(t))$$

Therefore, the derivative of V_2 can be obtain as follows:

$$\dot{V}_2 \leq \frac{p}{q}(1 - e^{-qh}) f^2(x(t)) - \frac{pqe^{-2qh}}{c^2(1 - e^{-qh})} f_1^2(x(t))$$

Adding $\dot{V}_1$ to $\dot{V}_2$ yields:

$$\dot{V} \leq b_1 x(t) f(x(t)) - qx(t) f_1(x(t)) + b_1 f(x(t)) f_1(x(t)) + \frac{p}{q}(1 - e^{-qh}) f^2(x(t)) - q\left(\frac{pe^{-2qh}}{c^2(1 - e^{-qh})} + 1\right) f_1^2(x(t))$$

If $b_1 < 0$ and $qc > 0$, then by applying the assumption $0 \leq \frac{f(y)}{y} \leq m$ yields:

$$\dot{V} \leq \left[\frac{b_1}{m} + \frac{p}{q}(1 - e^{-qh})\right] f^2(x(t)) + \left[b_1 - \frac{q}{m}\right] f(x(t)) f_1(x(t)) - q\left[\frac{pe^{-2qh}}{c^2(1 - e^{-qh})} + 1\right] f_1^2(x(t))$$

In order to satisfy the conditions $b_1 < 0$ and $qc > 0$, one has :

$$b_1 = b + ce^{qh} < 0 \quad \Rightarrow$$

$$\begin{cases} if \quad c > 0, b < 0 \;\Rightarrow\; e^{qh} < -\frac{b}{c} \\ \quad \Rightarrow\; 0 < q < \frac{1}{h}\ln\left(-\frac{b}{c}\right) \quad that \quad |c| < |b| \\ if \quad c < 0, b > 0 \;\Rightarrow\; e^{qh} > -\frac{b}{c} \\ \quad \Rightarrow\; 0 > q > \frac{1}{h}\ln\left(-\frac{b}{c}\right) \quad that \quad |c| > |b| \\ if \quad c < 0, b < 0 \;\Rightarrow\; q \in \Re^- \end{cases} \qquad (20)$$

One of the results which is derived from above relations is $b + c < 0$ that it is one of the assumptions of problem.

In order to be converted $\dot{V}$ to the quadratic form in terms of f or f_1, one should have the following inequalities [26]:

$$i) \quad q\left[\frac{pe^{-2qh}}{c^2(1 - e^{-qh})} + 1\right] \geq 0 \quad \Rightarrow \quad p \geq c^2(e^{qh} - e^{2qh})$$

$$ii) \quad \frac{b_1}{m} + \frac{p}{q}(1 - e^{-qh}) \leq 0 \quad \Rightarrow \quad p \leq \frac{qb_1}{m(e^{-qh} - 1)}$$

If the inequality (i) is satisfied, then the following inequality should be hold :

$$-q\left(\frac{pe^{-2qh}}{c^2(1-e^{-qh})}+1\right)f_1^2(x(t))+\left(b_1-\frac{q}{m}\right)f(x(t))f_1(x(t))$$

$$\leq \frac{\left(b_1-\frac{q}{m}\right)^2}{4q\left(\frac{pe^{-2qh}}{c^2(1-e^{-qh})}+1\right)}f^2(x(t))$$

finally, $\dot{V}$ is obtained as follows:

$$\dot{V} \leq \left[\frac{b_1}{m}+\frac{p}{q}\left(1-e^{-qh}\right)+\frac{c^2\left(b_1-\frac{q}{m}\right)^2\left(1-e^{qh}\right)}{4q\left[pe^{-2qh}+c^2\left(1-e^{-qh}\right)\right]}\right]f^2(x(t))$$

If $\dot{V}$ is negative definite (or the system (19) would be stable in the sense of Lyapunov), then :

$$\left[\frac{b_1}{m}+\frac{p}{q}\left(1-e^{-qh}\right)+\frac{c^2\left(b_1-\frac{q}{m}\right)^2\left(1-e^{qh}\right)}{4q\left[pe^{-2qh}+c^2\left(1-e^{-qh}\right)\right]}\right] \leq 0 \qquad (21)$$

From (i) and (ii), there exists p such that :

$$c^2\left(e^{qh}-e^{2qh}\right) \leq p \leq \frac{qb_1}{m\left(e^{-qh}-1\right)} \qquad (22)$$

$$c^2\left(e^{qh}-e^{2qh}\right) \leq \frac{qb_1}{m\left(e^{-qh}-1\right)} \qquad (23)$$

So, q is derived properly from (20) and (23) initially and then p will be determined by (22). Applying (22) and (23) in (21) yields [26]:

$$c^2\left(e^{qh}-1\right)+\frac{qb_1}{m(e^{qh}-1)}-|c|\left(b_1+\frac{q}{m}\right) \leq 0 \qquad (24)$$

The inequality (24) may be reduced to

$$(b+c)\left(-c+\frac{q}{m(e^{qh}-1)}\right)\leq 0 \qquad \textit{for } c>$$

$$\left(c\left(e^{2qh}-1\right)+b\right)\left(c+\frac{q}{m(e^{qh}-1)}\right)\leq 0 \qquad \textit{for } c\leq$$

(2

Obviously, derivation of the stability results from (2 is simpler than (21). For example, when $c>0$, accordi to the constraint of the problem ($b+c<0$), the relati (21) would be converted to :

$$-c+\frac{q}{m(e^{qh}-1)} \geq 0 \quad\Rightarrow\quad \frac{q}{e^{qh}-1} \geq mc$$

If the above relation can establish with the selected from (20) and (23), the amount of delay, the existence coefficients of the system and a proper m, then the system (19) will be Lyapunov stable. In other hands, the amount of m determines the boundary of local stability of the system.

Remark (3): An important note about the introduced method is that when $c<0$, one may derive independent-delay stability results of the system (19) letting $q\to-\infty$. It is shown that if $b\leq c$ then the system (19) is independent-delay Lyapunov stable [26].

5. The Numerical Example

Consider the following dynamic equation:

$$\dot{x}(t)=x^3(t)-20x^3(t-h) \quad , \quad t\geq 0 \qquad (26)$$

The initial conditions of the system is assumed to be as follows:

$$\varphi(t)=0.5e^{5t} \quad , \quad -h\leq t\leq 0$$

The object here is to investigate the effect of delay (h) on stability of system (26) and compare the simulation results with the proposed theory. By carrying out a simulation, one can observe that the system (26) would be stable up to $h=0.43$. For example, if $h=0.2$, we choose $q=-0.1$ and obtain the range of p as follows :

$7.76\leq p\leq 306.97$

Choosing $p=50$, the derivative of Lypunov functional would be as follows:

$$\dot{V} \leq \left[\frac{-18.604}{m}+10+0.454\left(-18.604+0.1\frac{1}{m}\right)^2\right]f^2(x(t))$$

If the value of m is within the interval $0<m\leq 0.3026$, then $\dot{V}$ is negative definite. In order to prove such a condition, the initial value of the state variable of the system (26) should be within the following region:

$$0<\frac{x^3}{x}\leq 0.3026 \quad\Rightarrow\quad x(0)\leq 0.56$$

The simulation results show that system (26) is locally stable with the above preset values.

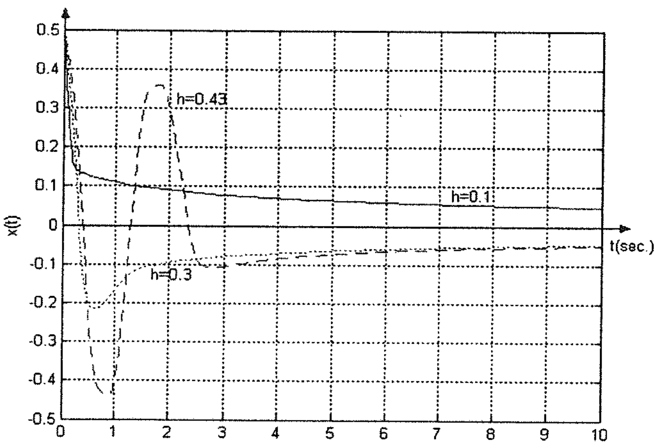

Figure 1 : The trajectories of system (26) with $h\leq 0.43$

6. CONCLUSION

In this paper, we extend the Lyapunov-Krasovskii theorems to some types of nonlinear time-delay systems by deriving a new algorithm for generating proper Lyapunov-Krasovskii functionals. These functionals can analyze delay-dependent stability for mentioned systems. Obviously, the obtained conditions are sufficient and used to analyze local stability. The proposed results give also less conservative conditions. By applying the introduced procedure to a typical system, the more results have been obtained so that, by solving a numerical example, the capability of the algorithm is demonstrated. So, it is possible to derive delay-independent stability conditions in this case. This method could be extended to a more general case where we have $\dot{x}(t) = BF(x(t)) + CF(x(t-h))$ and even to time varying nonlinear systems [26].

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