$\sum_{n=1}^{\infty} f_n$ is uniformly bounded. Hence, by theorem 2.9, the series $\sum_{n=1}^{\infty} g_n f_n$ is exponentially convergent, but it is not uniformly convergent on X.

(iii) For $n = 1,2,3,\ldots$, suppose that $f_n(x)=\frac{e^{ax}}{n^2}$ and $g_n(x)=\frac{x^3 e^{-ax}}{n}$, where $0 < a \leq r$. The sequence $\{g_n\}_{n=1}^{\infty}$ is monotonically decreasing and also uniformly bounded [$|g_n(x)| \leq \left(\frac{3}{ea}\right)^3$, for each $x \in X$ and $n = 1,2,3,\ldots$]. Hence, by example (i) and theorem 2.10, the series $\sum_{n=1}^{\infty} g_n f_n$ is exponentially convergent, but it is not uniformly convergent on X.

References

[1] T.M.Apostol, Mathematical Analysis, Addison-Wesley Publishing Company Inc. Second Edition,1974
[2] W.Rudin, Functional Analysis, McGraw-Hill,Inc. Second Edition,1991

$$\left\|\gamma^{r}_{F_n,F_m}\right\|_2 = d_{E_r}(F_n, F_m) < \frac{\varepsilon}{3M}, \tag{14}$$

[c.f. theorem 2.1.(ii)], where M is a uniform bound for $\{g_n\}_{n=1}^{\infty}$. Let $N = \max\{N_0, N_\varepsilon\}$ and $S_n(x) = \sum_{k=1}^{n} g_k(x) f_k(x)$. Now, by using Abel's lemma in [1], for $n > m \geq N$ we have

$$\|S_n(x) - S_m(x)\| = \left\| g_{n+1}(x) F_n(x) - g_{m+1}(x) F_m(x) + \sum_{k=m+1}^{n} (g_k(x) - g_{k+1}(x)) F_k(x) \right\|$$

$$= \left\| g_{n+1}(x) F_{m,n}(x) + \sum_{k=m+1}^{n} (g_k(x) - g_{k+1}(x)) F_{m,k}(x) \right\|$$

$$\leq M \|F_{m,n}(x)\| + \sum_{k=m+1}^{n} (g_k(x) - g_{k+1}(x)) \|F_{m,k}(x)\|$$

$$= M \|F_{m,n}(x)\| + g_{m+1}(x) \|F_{m,m+1}(x)\| + \sum_{k=m+2}^{n} g_k(x) \|F_{m,k}(x)\|$$

$$- g_{n+1}(x) \|F_{m,n}(x)\| - \sum_{k=m+1}^{n-1} g_{k+1}(x) \|F_{m,k}(x)\|$$

$$\leq 2M \|F_{m,n}(x)\| + M \|F_{m,m+1}(x)\| + M \sum_{k=m+2}^{n} \left(\|F_{m,k}(x)\| - \|F_{m,k-1}(x)\| \right)$$

$$= 3M \|F_{m,n}(x)\| \leq \left(3M\gamma^{r}_{F_m,F_n}, x^{(r,\xi)}\right).$$

Thus, by (14) and definition 1.3, we have

$$d_{E_r}(S_n, S_m) \leq \left\| 3M\gamma^{r}_{F_m,F_n} \right\|_2 < \varepsilon.$$

Now, use theorem 2.5.

Examples 2.11. In the following examples, consider $X = [0, +\infty)$, $Y = (-\infty, +\infty)$ and $\xi = \eta = 0$.

(i) For $n = 1,2,3,\ldots$, suppose that $f_n(x) = \frac{e^{ax}}{n^2}$, where $|a| \leq r$. Consider γ as in example 1.2.(v), let $\gamma_n = \frac{1}{n^2}\gamma$, then for each $x \in X$ and $n = 1,2,3,\ldots$, we have $|f_n(x)| \leq (\gamma_n, x^{(r,0)})$, so that each f_n is exponentially bounded. Furthermore, $\|f_n\|_{E_r} = d_{E_r}(f_n, 0) \leq \frac{1}{n^2}\|\gamma\|_2$. Thus, $\sum_{n=1}^{\infty} \|f_n\|_{E_r}$ is convergent. Hence, by theorem 2.8, the series $\sum_{n=1}^{\infty} \frac{e^{ax}}{n^2}$ is exponentially convergent, but it is not uniformly convergent on X.

(ii) For $n = 1,2,3,\ldots$, suppose that $f_n(x) = \frac{1}{n^2}$ and $g_n(x) = \frac{x^3}{n}$. The sequence $\{g_n\}_{n=1}^{\infty}$ is monotonically decreasing and since, $d_{E_r}(g_n, 0) \leq \frac{1}{n}$, so that $g_n \to 0$ exponentially. The series

Now, use theorem 2.5.

Theorem 2.9. (*Dirichlet's test*) Let X the a metric space and Y a Banach space. Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions from X to Y. Let $F_n = \sum_{k=1}^{n} f_k$ and assume that $\{F_n\}_{n=1}^{\infty}$ is uniformly bounded on X. Let $\{g_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on X, such that $g_{n+1}(x) \le g_n(x)$, for each $x \in X$ and $n = 1,2,3,\ldots$, and assume that $g_n \to 0$ exponentially. Then the series $\sum_{n=1}^{\infty} g_n f_n$ is exponentially convergent on X.

Proof. Since $g_n \to 0$ exponentially, then by theorem 2.1.(ii), we have $\left\|\gamma^r_{g_n,0}\right\|_2 \to 0$, so that for $\varepsilon > 0$ there is the natural number N_ε, such that for all $n \ge N_\varepsilon$,

$$\left\|\gamma^r_{g_n,0}\right\|_2 = d_{E_r}(g_n,0) < \frac{\varepsilon}{2M}, \tag{13}$$

where M is a uniform bound for $\{F_n\}_{n=1}^{\infty}$. Let $S_n(x) = \sum_{k=1}^{n} g_k(x) f_k(x)$. Now, by using Abel's lemma in [1], for $n > m \ge N_\varepsilon$ we have

$$\|S_n(x) - S_m(x)\| = \left\| g_{n+1}(x)F_n(x) - g_{m+1}(x)F_m(x) + \sum_{k=m+1}^{n} (g_k(x) - g_{k+1}(x))F_k(x) \right\|$$

$$\le M g_{n+1}(x) + M g_{m+1}(x) + M \sum_{k=m+1}^{n} (g_k(x) - g_{k+1}(x))$$

$$= 2M g_{m+1}(x) \le \left(2M\gamma^r_{g_{m+1},0}, x^{(r,\xi)}\right).$$

Thus, by (13) and definition 1.3, for $n > m \ge N_\varepsilon$, we have

$$d_{E_r}(S_n, S_m) \le \left\|2M\gamma^r_{g_{m+1},0}\right\|_2 < \varepsilon.$$

Now, use theorem 2.5.

Theorem 2.10. (*Abel's test*) Let X be a metric space and Y a Banach space. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions from X to Y, such that $\sum_{n=1}^{\infty} f_n$ is exponentially convergent on X. Suppose that there is a natural number N_0 such that

$$\left\|\sum_{k=m}^{n} f_k(x)\right\| \le \left\|\sum_{k=m}^{n+1} f_k(x)\right\|,$$

for all $x \in X$ and $n > m \ge N_0$. Let $\{g_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on X, such that $g_{n+1}(x) \le g_n(x)$, for each $x \in X$ and $n = 1,2,3,\ldots$. Assume that $\{g_n\}_{n=1}^{\infty}$ is uniformly bounded on X. Then the series $\sum_{n=1}^{\infty} g_n f_n$ is exponentially convergent on X.

Proof. Suppose that $F_n = \sum_{k=1}^{n} f_k$ and $F_{m,n} = F_n - F_m$. By assumption, for $\varepsilon > 0$ there is the natural number N_ε such that for $n > m \ge N_\varepsilon$, and $d(x_0,\xi)\le s$, we have

$$d(f(x),f(x_0))\le d(f(x),f_{N_\varepsilon}(x))+d(f_{N_\varepsilon}(x),f_{N_\varepsilon}(x_0))+d(f_{N_\varepsilon}(x_0),f(x_0))$$

$$<d_{E_r}(f,f_{N_\varepsilon})\exp\left(\frac{r^2}{2}(d(x,\xi))^2\right)+\frac{\varepsilon}{3}+d_{E_r}(f,f_{N_\varepsilon})\exp\left(\frac{r^2}{2}(d(x_0,\xi))^2\right)$$

$$\le 2d_{E_r}(f,f_{N_\varepsilon})\exp\left(\frac{r^2}{2}s^2\right)+\frac{\varepsilon}{3}<\varepsilon.$$

Since $x_0\in X$ was arbitrary, we conclude that f is continuous on X.

Theorem 2.7. Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions on the real closed interval $[a,b]$ then the following assertions hold.

(i) Let α be a monotonically increasing function on the closed interval $[a,b]$. Suppose for $n=1,2,3,\dots$ the function f_n is integrable with respect to α on $[a,b]$, and $f_n\to f$ exponentially on $[a,b]$. Then f is integrable with respect to α on $[a,b]$, and

$$\int_a^b f\,d\alpha=\lim_n\int_a^b f_n\,d\alpha$$

(ii) Suppose for $n=1,2,3,\dots$, the function f_n is differentiable on $[a,b]$, and $\{f_n(c)\}_{n=1}^{\infty}$ converges, for some point c in $[a,b]$. If $\{f_n'\}_{n=1}^{\infty}$ converges exponentially on $[a,b]$, then $\{f_n\}_{n=1}^{\infty}$ converges exponentially on $[a,b]$ to a function f, and

$$f'(x)=\lim_n f_n'(x)\qquad (a\le x\le b).$$

Proof. Use theorem 2.2.

Theorem 2.8. *(Weierstrass M-test)* Let X is a metric space and Y is a Banach space. Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of exponentially bounded functions on X into Y, such that $\sum_{n=1}^{\infty}\|f_n\|_{E_r}$ is convergent, where $\|.\|_{E_r}=d_{E_r}(.,0)$. Then $\sum_{n=1}^{\infty}f_n$ is exponentially convergent on X.

Proof. Let $S_n=\sum_{k=1}^{n}f_k$, then by theorem 2.1.(ii), for each $x\in X$ and $n>m$, we have

$$\|S_n(x)-S_m(x)\|\le\sum_{k=m+1}^{n}\|f_k(x)\|=\sum_{k=m+1}^{n}d(f_k(x),0)$$

$$\le\sum_{k=m+1}^{n}\left(\gamma_{f_k,0}^{r},x^{(r,\xi)}\right)=\left(\left(\sum_{k=m+1}^{n}\gamma_{f_k,0}^{r}\right),x^{(r,\xi)}\right).$$

Thus, by definition 3.1 and theorem 2.1.(ii), we have

$$d_{E_r}(S_n,S_m)\le\left\|\sum_{k=m+1}^{n}\gamma_{f_k,0}^{r}\right\|_2\le\sum_{k=m+1}^{n}\|f_k\|_{E_r}.$$

$$\gamma_0 = \gamma^r_{f_j, f_{N_1}},$$
$$\gamma_m = \gamma^r_{f_{N_m}, f_{N_{m+1}}} \qquad m = 1,2,\ldots,K(n)-1,$$
$$\gamma_{K(n)} = \gamma^r_{f_{N_{K(n)}}, f_n}.$$

Then for $n \geq j \geq N_1$, we have

$$\begin{aligned} d(f_j(x), f_n(x)) &\leq d(f_j(x), f_{N_1}(x)) + d(f_{N_1}(x), f_{N_2}(x)) + \cdots \\ &\quad + d(f_{N_{K(n)-1}}(x), f_{N_{K(n)}}(x)) + d(f_{N_{K(n)}}(x), f_n(x)) \\ &\leq (\gamma_0, x^{(r,\xi)}) + (\gamma_1, x^{(r,\xi)}) + \cdots + (\gamma_{K(n)-1}, x^{(r,\xi)}) + (\gamma_{K(n)}, x^{(r,\xi)}) \\ &= \left(\sum_{m=0}^{K(n)} \gamma_m, x^{(r,\xi)} \right). \end{aligned}$$

Thus, for $j \geq N_1$ and each $x \in X$, we have

$$d(f_j(x), f(x)) = \lim_n d(f_j(x), f_n(x)) \leq (\gamma, x^{(r,\xi)}),$$

where $\gamma = \sum_{m=0}^{\infty} \gamma_m \in \ell^2$. Hence, for $j \geq N_1$, we have

$$d_{E_r}(f_j, f) \leq \|\gamma\|_2 \leq \sum_{m=0}^{\infty} \|\gamma_m\|_2 \leq \frac{\varepsilon}{2} + \sum_{m=1}^{\infty} 2^{-m} \varepsilon < 2\varepsilon.$$

This implies that $\{f_j\}_{j=1}^{\infty}$ converges to f exponentially.

Theorem 2.5. *(Cauchy condition)* Suppose Y is a complete space. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions defined on X into Y. Then $\{f_n\}_{n=1}^{\infty}$ is exponentially convergent, if and only if, it is exponentially Cauchy sequence.

Proof. Similar to proof of theorem 2.4.

Theorem 2.6. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions defined on X into Y. Let $f_n \to f$ exponentially, then f is continuous.

Proof. Fix $x_0 \in X$, put $s = \sup\{d(x,\xi) : d(x, x_0) < 1\}$. For $\varepsilon > 0$ there is the natural number N_ε such that if $n \geq N_\varepsilon$ then

$$d_{E_r}(f_n, f) < \frac{\varepsilon}{3} \exp\left(-\frac{r^2}{2} s^2 \right). \tag{11}$$

Since f_{N_ε} is continuous, there is $0 < \delta < 1$ such that if $d(x, x_0) < \delta$ then

$$d(f_{N_\varepsilon}(x), f_{N_\varepsilon}(x_0)) < \frac{\varepsilon}{3}. \tag{12}$$

Thus, for $x \in X$ such that $d(x, x_0) < \delta$; by (11), (12), theorem 2.1.(iii) and the fact that $d(x,\xi) \leq s$

$$d(f_n(x), f_m(x)) \le d_{E_r}(f_n, f_m) \exp\left(\frac{r^2}{2}(d(x,\xi))^2\right) < \varepsilon,$$

thus, for each $x \in X, \{f_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence in Y. By completeness of Y, $\{f_n(x)\}_{n=1}^{\infty}$ is convergent.

Define $f : X \to Y$, by $f(x) = \lim_n f_n(x)$. We claim that $f \in EB_r(\xi, \eta, X, Y)$ and $\{f_n\}_{n=1}^{\infty}$ converges to f exponentially.

For $\varepsilon > 0$, suppose $\{N_m\}_{m=1}^{\infty}$ is a sequence in the set of the natural numbers, such that

(i) $N_m < N_{m+1}$ for $m = 1,2,3,\dots$.

(ii) If $j \ge N_m$ then $d_{E_r}(f_j, f_{N_m}) < 2^{-m}\varepsilon$.

For $n \ge N_1$, let

$K(n) = \max\{m : N_m \le n\}$,

$\gamma_0 = \gamma^r_{f_{N_1}}$,

$\gamma_m = \gamma^r_{f_{N_{m+1}}, f_{N_m}} \qquad m = 1,2,\dots,K(n)-1,$

$\gamma_{K(n)} = \gamma_{f_n, f_{N_{K(n)}}}$,

(c.f. theorem 2.1.(ii)). We have

$$\|\gamma_m\|_2 < 2^{-m}\varepsilon \qquad m = 1,2,\dots,K(n) \text{ and } n \ge N_1 \tag{10}$$

Thus, for each $x \in X$ and $n \ge N_1$, we have

$$\begin{aligned} d(f_n(x), \eta) &\le d(f_n(x), f_{N_{K(n)}}(x)) + d(f_{N_{K(n)}}(x), f_{N_{K(n)-1}}(x)) \\ &\quad + \dots + d(f_{N_2}(x), f_{N_1}(x)) + d(f_{N_1}(x), \eta) \\ &\le (\gamma_{K(n)}, x^{(r,\xi)}) + (\gamma_{K(n)-1}, x^{(r,\xi)}) + \dots + (\gamma_1, x^{(r,\xi)}) + (\gamma_0, x^{(r,\xi)}) \\ &= \left(\sum_{m=0}^{K(n)} \gamma_m, x^{(r,\xi)}\right). \end{aligned}$$

But by (10), we have

$$\sum_{m=0}^{\infty} \|\gamma_m\|_2 \le \|\gamma_0\|_2 + \sum_{m=1}^{\infty} 2^{-m}\varepsilon = \|\gamma_0\|_2 + \varepsilon,$$

and since, ℓ^2 is a Banach space, so $\sum_{m=0}^{\infty} \gamma_m$ converges to an element $\gamma \in \ell^2$. Thus, for all $x \in X$, we have

$$d(f(x), \eta) = \lim_n d(f_n(x), \eta) \le \left(\sum_{m=0}^{\infty} \gamma_m, x^{(r,\xi)}\right) = (\gamma, x^{(r,\xi)}).$$

Hence, $f \in EB_r(\xi, \eta, X, Y)$.

Similarly for $n \ge j \ge N_1$, let

Hence, $f_n \to f$ uniformly.

Note 2.3. By theorem 2.2, we have the following implications:
uniformly convergence $\Rightarrow$ exponentially convergence $\Rightarrow$ pointwise convergence.
In what follows, we show that the converse of these implications are not true in general.
(i) Let f be a function defined on the set of the real numbers, by

$$f(x)=\begin{cases} x-1 & \textbf{if x is integer} \\ [x] & \textbf{otherwise.} \end{cases}$$

Where $[x]$ denotes the integral part of the real number x. For $n=1,2,3,\ldots$, let $f_n(x)=\left[x-\frac{1}{n}\right]$, for each real number x. Since, for each real number x we have $d(f(x),0)=|f(x)|\le 1+|x|$, and $d(f_n(x),0)=|f_n(x)|\le 1+|x|$, for $n=1,2,3,\ldots$; then by definition 1.1 the functions $f, f_1, f_2, \ldots$ are exponentially bounded of rank $r\neq 0$ relative to $(0,0)$. Moreover $f_n(x)\to f(x)$ for each real number x. But $\{f_n\}_{n=1}^{\infty}$ does not converge to f exponentially. Since by theorem 2.1.(iii), we have

$$|f_n(x)-f(x)|=d(f_n(x),f(x))\le d_{E_r}(f_n,f)\exp\left(\frac{r^2}{2}x^2\right).$$

For $n>1$, let $x=\frac{n^2+1}{n^2}$, then

$$d_{E_r}(f_n,f)\ge\left|\left[\frac{n^2+1}{n^2}-\frac{1}{n}\right]-\left[\frac{n^2+1}{n^2}\right]\right|\exp\left(-\frac{r^2}{2}\left(\frac{n^2+1}{n^2}\right)^2\right)=\exp\left(-\frac{r^2}{2}\left(\frac{n^2+1}{n^2}\right)^2\right)$$

$$>\exp(-r^2),$$

which shows that $\{f_n\}_{n=1}^{\infty}$ does not converge to f exponentially.

(ii) Consider the sequence $f_n(x)=x/n$ defined on the set of the real numbers, $\{f_n\}_{n=1}^{\infty}$ is a sequence of exponentially bounded functions of rank $r\neq 0$ relative to $(0,0)$. For $\gamma_n=(0,1/rn,0,0,\ldots)$ we have $d(f_n(x),0)=|x/n|=\left(\gamma_n, x^{(r,0)}\right)$, thus

$$d_{E_r}(f_n,0)\le\|\gamma_n\|_2=\frac{1}{rn}\to 0 \qquad \text{as } n\to\infty.$$

Hence, $f_n\to 0$ exponentially. But $\{f_n\}_{n=1}^{\infty}$ does not converge uniformly to zero.

Theorem 2.4. If Y is a complete space, then $EB_r(\xi,\eta,X,Y)$ is a complete space.

Proof. Suppose Y is a complete space. Let $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $EB_r(\xi,\eta,X,Y)$ then for each $x\in X$, there is the natural number N_x, such that $d_{E_r}(f_n,f_m)<\varepsilon\exp\left(-\frac{r^2}{2}(d(x,\xi))^2\right)$, for all $n\ge m\ge N_x$. So, by theorem 2.1.(iii), for all $n\ge m\ge N_x$, we have

$$d(f(x),g(x)) = t(d(f(x),g(x))) + (1-t)d(f(x),g(x)) \le t(\gamma, x^{(r,\xi)}) + (1-t)(\gamma', x^{(r,\xi)})$$
$$= ((t\gamma + (1-t)\gamma'), x^{(r,\xi)}),$$

for all $x \in X$; hence, $(t\gamma + (1-t)\gamma') \in \Gamma_r(f,g)$.

Suppose γ is in the closure of $\Gamma_r(f,g)$. Let $\{\gamma_n\}_{n=1}^{\infty}$ be a sequence in $\Gamma_r(f,g)$ such that $\gamma_n \to \gamma$. Since $d(f(x),g(x)) \le (\gamma_n, x^{(r,\xi)})$, for all $x \in X$ and $n = 1,2,3,\ldots$; therefore,

$$d(f(x),g(x)) \le \lim_n (\gamma_n, x^{(r,\xi)}) = \left(\lim_n \gamma_n, x^{(r,\xi)}\right) = (\gamma, x^{(r,\xi)}).$$

Hence, $\gamma \in \Gamma_r(f,g)$. Proof for $\Gamma_r(f)$ is similar.

(ii) Existence and uniqueness of γ_f^r and $\gamma_{f,g}^r$ follows from theorem 12.3 in [2].

(iii) By (ii), definition 1.1 and Schwarz' inequality, we have

$$d(f(x),g(x)) \le (\gamma_{f,g}^r, x^{(r,\xi)}) \le \|\gamma_{f,g}^r\|_2 \|x^{(r,\xi)}\|_2 = d_{E_r}(f,g)\|x^{(r,\xi)}\|_2$$

$$\le d_{E_r}(f,g)\exp\left(\frac{r^2}{2}(d(x,\xi))^2\right).$$

Proof of (8) is similar.

We say that the sequence $\{f_n\}_{n=1}^{\infty}$ converges to f exponentially (or $f_n \to f$ exponentially), if $\{f_n\}_{n=1}^{\infty}$ converges to f in exponentially metric (for fixed r, ξ and η).

Theorem 2.2. Suppose $f, f_1, f_2, \ldots$ are functions from X into Y.

(i) If $\{f_n\}_{n=1}^{\infty}$ converges to f uniformly, then it converges to f exponentially.

(ii) If $\{f_n\}_{n=1}^{\infty}$ converges to f exponentially, then it converges to f pointwise.

(iii) If X is bounded and, if $\{f_n\}_{n=1}^{\infty}$ converges to f exponentially, then it converges to f uniformly.

Proof. (i) For $\varepsilon > 0$ there is a natural number N_ε, such that $d(f_n(x),f(x)) < \varepsilon/2$, for all $x \in X$ and $n \ge N_\varepsilon$. Let $\gamma = (\varepsilon/2, 0, 0, \ldots)$, then $d(f_n(x),f(x)) < \varepsilon/2 = (\gamma, x^{(r,\xi)})$, for all $x \in X$ and $n \ge N_\varepsilon$. So by definition of $\Gamma_r(f_n, f)$, we have

$$d_{E_r}(f_n, f) \le \|\gamma\|_2 < \varepsilon \text{ for all } n \ge N_\varepsilon .$$

(ii) Suppose $d_{E_r}(f_n,f) \to 0$, then by theorem 2.1.(iii), for each $x \in X$, we have

$$d(f_n(x),f(x)) \le d_{E_r}(f_n,f)\exp\left(\frac{r^2}{2}(d(x,\xi))^2\right) \to 0 \text{ as } n \to \infty.$$

Hence, $f_n(x) \to f(x)$, for all $x \in X$.

(iii) For $\varepsilon > 0$ there is a natural number N_ε, such that $d_{E_r}(f_n,f) < \varepsilon\exp\left(-\frac{r^2}{2}s^2\right)$, where $s = \sup\{d(x,\xi) : x \in X\}$, thus by theorem 2.1.(iii), for all $x \in X$ and $n \ge N_\varepsilon$, we have

$$d(f_n(x),f(x)) \le d_{E_r}(f_n,f)\exp\left(\frac{r^2}{2}(d(x,\xi))^2\right) \le d_{E_r}(f_n,f)\exp\left(\frac{r^2}{2}s^2\right) < \varepsilon.$$

$d_{E_r}(f,g)=0$, then by definition 1.3, there is a sequence $\{\gamma_n\}_{n=1}^{\infty}$ in $\Gamma_r(f,g)$ such that $\|\gamma_n\|_2 \to d_{E_r}(f,g)=0$ and $d(f(x),g(x)) \le (\gamma_n, x^{(r,\xi)})$, for all $x \in X$ and $n = 1,2,3,\ldots$, thus

$$d(f(x),g(x)) \le \lim_n (\gamma_n, x^{(r,\xi)}) = \left(\lim_n \gamma_n, x^{(r,\xi)}\right) = (0, x^{(r,\xi)}) = 0,$$

hence, $f = g$.

$d_{E_r}(f,g) = d_{E_r}(g,f)$ since $\Gamma_r(f,g) = \Gamma_r(g,f)$.

Finally for $f,g,h \in EB_r(\xi,\eta,X,Y)$ by definition 1.3, there are sequences $\{\gamma_n\}_{n=1}^{\infty}$ in $\Gamma_r(f,h)$ and $\{\gamma'_n\}_{n=1}^{\infty}$ in $\Gamma_r(h,g)$ such that $\|\gamma_n\|_2 \to d_{E_r}(f,h)$ and $\|\gamma'_n\|_2 \to d_{E_r}(h,g)$. Thus

$$d(f(x),g(x)) \le d(f(x),h(x)) + d(h(x),g(x)) \le (\gamma_n, x^{(r,\xi)}) + (\gamma'_n, x^{(r,\xi)}) = (\gamma_n + \gamma'_n, x^{(r,\xi)}),$$

for all $x \in X$ and $n = 1,2,3,\ldots$, therefore $(\gamma_n + \gamma'_n) \in \Gamma_r(f,g)$, and using definition 1.3, once more we obtain

$$d_{E_r}(f,g) \le \|\gamma_n + \gamma'_n\|_2 \le \|\gamma_n\|_2 + \|\gamma'_n\|_2,$$

for $n = 1,2,3,\ldots$; hence

$$d_{E_r}(f,g) \le \lim_n \left(\|\gamma_n\|_2 + \|\gamma'_n\|_2\right) = \lim_n \|\gamma_n\|_2 + \lim_n \|\gamma'_n\|_2 = d_{E_r}(f,h) + d_{E_r}(h,g).$$

2-Main Results

Theorem 2.1. For $f,g \in EB_r(\xi,\eta,X,Y)$, the following assertions hold.

(i) The sets $\Gamma_r(f)$ and $\Gamma_r(f,g)$ are closed convex subsets of ℓ^2.

(ii) There are unique elements $\gamma_f^r \in \Gamma_r(f)$ and $\gamma_{f,g}^r \in \Gamma_r(f,g)$ such that

$$\delta_{E_r}(f) = \|\gamma_f^r\|_2, \tag{6}$$

$$d_{E_r}(f,g) = \|\gamma_{f,g}^r\|_2. \tag{7}$$

(iii) For all $x \in X$,

$$d(f(x),\eta) \le \delta_{E_r}(f) \exp\left(\frac{r^2}{2}(d(x,\xi))^2\right), \tag{8}$$

$$d(f(x),g(x)) \le d_{E_r}(f,g) \exp\left(\frac{r^2}{2}(d(x,\xi))^2\right) \tag{9}$$

In general the inequalities (7) and (9) are true, if f and g are not exponentially bounded, and $\Gamma_r(f,g)$ is not empty.

Proof. (i) Let $\gamma,\gamma' \in \Gamma_r(f,g)$ and $0 < t < 1$, by definition 1.3, we have $d(f(x),g(x)) \le (\gamma, x^{(r,\xi)})$ and $d(f(x),g(x)) \le (\gamma', x^{(r,\xi)})$, for all $x \in X$, thus

$$\frac{\delta}{2}=k-(k-\delta/2)\le\alpha-(k-\delta/2)=\|x_1\|-\|x_2\|. \qquad (5)$$

Now, since $x_1\in A$ and $x_2\notin A$, by (5), we obtain

$$\|f(x_1)-f(x_2)\|\ge\|f(x_1)\|-\|f(x_2)\|>(a+b\|x_1\|)-(a+b\|x_2\|)=b(\|x_1\|-\|x_2\|)\ge b\frac{\delta}{2}=\varepsilon,$$

which contradicts (1) and (4). Hence A is empty.

(iv) Suppose $X=Y$ is a normed algebra over the field of the complex numbers. Consider $EB_r(X)=EB_r(0,0,X,X)$. Let $P(x)=\sum_{j=0}^{n}a_jx^j$ be a polynomial on X, then $P\in\bigcap_{r\neq0}EB_r(X)$. Because, we may choose

$$\gamma=\left(|a_0|,\frac{|a_1|}{r},\frac{2|a_2|}{r^2},\frac{6|a_3|}{r^3},\frac{6|a_4|}{r^4},\ldots,\frac{(n-1)!|a_n|}{r^n},0,0,\ldots\right).$$

Note that in this case, $d(.,0)=\|.\|$. In particular, if Π is the set of all polynomials on X then $\overline{\Pi}\subseteq\bigcap_{r\neq0}EB_r(X)$ [in uniform norm]. To see this, let $f\in\overline{\Pi}$ then there is $P\in\Pi$ such that $\|f(x)-P(x)\|<1$, for all $x\in X$, thus $\|f(x)\|<\|P(x)\|+1$, for all $x\in X$. Let $P(x)=\sum_{j=0}^{n}a_jx^j$, then for f, put

$$\gamma=\left(|a_0|+1,\frac{|a_1|}{r},\frac{2|a_2|}{r^2},\frac{6|a_3|}{r^3},\frac{6|a_4|}{r^4},\ldots,\frac{(n-1)!|a_n|}{r^n},0,0,\ldots\right).$$

(v) Suppose $X=Y$ is a Banach algebra over the field of the complex numbers. Consider $EB_r(X)=EB_r(0,0,X,X)$. Let $\exp(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ then for $a\in X$, the function $\exp(ax)$ is in $EB_r(X)$ for all $r\ge\|a\|$; put $\gamma=(1,1,1,1,1/4,1/5,\ldots,1/n,\ldots)$.

In particular, if $f:X\to X$, is a function such that for all $x\in X$, $\|f(x)\|\le\|\exp(ax)\|$ for some $a\in X$, then f is in $EB_r(X)$ for all $r\ge\|a\|$ (this modifies the terminology).

Definition 1.3. For $f,g\in EB_r(\xi,\eta,X,Y)$, define

$\Gamma_r(f)=\{\gamma\in\ell^2:d(f(x),\eta)\le(\gamma,x^{(r,\xi)})\ \forall x\in X\}$,

$\Gamma_r(f,g)=\{\gamma\in\ell^2:d(f(x),g(x))\le(\gamma,x^{(r,\xi)})\ \forall x\in X\}$,

$\delta_{E_r}(f)=\inf\{\|\gamma\|_2:\gamma\in\Gamma_r(f)\}$,

$d_{E_r}(f,g)=\inf\{\|\gamma\|_2:\gamma\in\Gamma_r(f,g)\}$.

Note that $\Gamma_r(f,g)$ is not empty, since for $\gamma_1\in\Gamma_r(f)$ and $\gamma_2\in\Gamma_r(g)$ we have

$d(f(x),g(x))\le d(f(x),\eta)+d(g(x),\eta)\le(\gamma_1,x^{(r,\xi)})+(\gamma_2,x^{(r,\xi)})=(\gamma_1+\gamma_2,x^{(r,\xi)})$,

hence, $\Gamma_r(f)+\Gamma_r(g)\subseteq\Gamma_r(f,g)$.

Moreover, $d_{E_r}(f,g)$ makes sense whenever, f and g are not exponentially bounded, and $\Gamma_r(f,g)$ is not empty.

Theorem 1.4. The set $EB_r(\xi,\eta,X,Y)$ under the distance d_{E_r} in definition 1.3 is a metric space. The metric d_{E_r} is called **Exponentially Metric.**

Proof. Let $f,g\in EB_r(\xi,\eta,X,Y)$, it is trivial that $d_{E_r}(f,g)\ge0$ and, $d_{E_r}(f,g)=0$ if $f=g$. Suppose

Examples 1.2. Consider the function $f: X \to Y$.

(i) If f is bounded, i.e. $\operatorname{diam} f(X) < \infty$, then it is exponentially bounded of rank r relative to (ξ,η) for all real number r, and each $\xi \in X$ and $\eta \in Y$. To see this put $d = \operatorname{diam} f(X) + \operatorname{dist}(\eta, f(X))$, and $\gamma = (d,0,0,\ldots)$ in definition 1.1. In particular, each constant function f is in $EB_r(\xi,\eta,X,Y)$.

(ii) Let $g \in EB_r(\xi,\eta,X,Y)$. Let $d(f(x),\eta) \le d(g(x),\eta)$ for all $x \in X$, then $f \in EB_r(\xi,\eta,X,Y)$.

(iii) Suppose X and Y are normed linear spaces over the field of the complex numbers. If f is uniformly continuous, then it is exponentially bounded of rank r relative to (ξ,η) for all $r \neq 0$, $\xi \in X$ and $\eta \in Y$. To see this, first we claim that for uniformly continuous function $f: X \to Y$, there are $a, b \ge 0$, such that

$$\|f(x)\| \le a + b\|x\| \quad \text{for all } x \in X.$$

Thus,

$$d(f(x),\eta) = \|f(x) - \eta\| \le \|\eta\| + a + b\|\xi\| + b\|x - \xi\| \quad \text{for all } x \in X.$$

Now, take γ to be $(a + \|\eta\| + b\|\xi\|, b/r, 0, 0, \ldots)$ in definition 1.1. To prove our claim, consider a uniformly continuous function f. Without loss of generality, assume that $f(0) = 0$ [Otherwise define a new function $g: X \to Y$ by $g(x) = f(x) - f(0)$].

For $\varepsilon > 0$ there exists $\delta > 0$ such that for $x_1, x_2 \in X$,

$$\|x_1 - x_2\| < \delta \Rightarrow \|f(x_1) - f(x_2)\| < \varepsilon. \tag{1}$$

Put $a = \varepsilon$, $b = \dfrac{2\varepsilon}{\delta}$ and consider $x \in X$. If $\|x\| < \delta$, then by (1) we have

$$\|f(x)\| = \|f(x) - f(0)\| < \varepsilon \le a + b\|x\|.$$

Put $A = \{x \in X : \|x\| \ge \delta \text{ and } \|f(x)\| > a + b\|x\|\}$. We claim that A is empty. Assume on contrary that A is nonempty, then the set $B = \{\|x\| : x \in A\}$ is also nonempty}. Let $k = \inf B$ and consider $\alpha \in B$ and $x_1 \in X$, such that

$$0 < \delta \le k \le \alpha < k + \frac{\delta}{2} \quad \text{and} \quad \|x_1\| = \alpha. \tag{2}$$

Let $t = \dfrac{k - \delta/2}{\alpha}$ and $x_2 = t x_1$. Since $0 < t < 1$ we have

$$\|x_2\| = t\|x_1\| = t\alpha = k - \frac{\delta}{2} < k, \tag{3}$$

thus, $x_2 \notin A$. On the other hand, by (2) we have

$$\|x_1 - x_2\| = (1 - t)\alpha = \alpha - (k - \delta/2) < (k + \delta/2) - (k - \delta/2) = \delta \tag{4}$$

and also, by (2) and (3), we have

Exponentially Metric on Function Spaces

B. Behdin
Ph. D. Student

A. Riazi
Professor

Faculty of Mathematics and Computer,
AmirKabir University of Technology

Abstract

In this paper a new metric (called Exponentially Metric) is introduced on functions from a metric space X into another metric space Y. Exponentially metric can be defined on a large set of functions (called Exponentially Bounded Functions) enjoying interesting properties similar to uniform metric. Some results concerning the exponentially meric are proved.

Keywords

Metric spaces; Function spaces; Exponential function; Convergence of sequences; Divergent series

1- Preliminaries and definitions

Throughout this paper X and Y are metric spaces, the symbol (γ_1,γ_2) denotes the inner product of two elements γ_1 and γ_2 in the real Hilbert space ℓ^2.

Definition 1.1. For fixed real number r, fixed $\xi\in X$ and arbitrary $x\in X$, consider the sequence $x^{(r,\xi)}=\{x_n^{(r,\xi)}\}_{n=0}^{\infty}$ defined by,

$$x_n^{(r,\xi)}=\begin{cases}\dfrac{r^n(d(x,\xi))^n}{n!} & n=0,1,2,3\\ \dfrac{r^n(d(x,\xi))^n}{(n-1)!} & n=4,5,\ldots\end{cases}$$

By using definition of norm $\|\cdot\|_2$ in ℓ^2 and the fact that $((n-1)!)^2>n!$ for $n=4,5,6,\ldots$, a simple calculation shows that $1\le\left\|x^{(r,\xi)}\right\|_2\le\exp\left(\frac{r^2}{2}(d(x,\xi))^2\right)$, hence, $x^{(r,\xi)}$ is a nonzero element of ℓ^2.

For fixed $\eta\in Y$, the function $f:X\to Y$, is called **Exponentially Bounded** of rank r relative to (ξ,η), if there exists $\gamma\in\ell^2$ such that

$$d(f(x),\eta)\le(\gamma,x^{(r,\xi)})\quad\text{for all}\quad x\in X.$$

We denote the set of all exponentially bounded functions of rank r relative to (ξ,η), from X to Y by $EB_r(\xi,\eta,X,Y)$.

For $\gamma=\{c_n\}_{n=0}^{\infty}$ in ℓ^2, define $|\gamma|=\{|c_n|\}_{n=0}^{\infty}$. We know that $|\gamma|\in\ell^2$. If $f\in EB_r(\xi,\eta,X,Y)$, then for $0\le r\le s$ we have

$$d(f(x),\eta)\le\left|(\gamma,x^{(r,\xi)})\right|\le(|\gamma|,x^{(r,\xi)})\le(|\gamma|,x^{(s,\xi)}),$$

for all $x\in X$. Hence, $EB_r(\xi,\eta,X,Y)\subseteq EB_s(\xi,\eta,X,Y)$ if $0\le r\le s$.