Tabl (1) **b = 10**

K	Re (λ_k)	I_m (λ_k)
1	-7.6657	-.3449
2	-5.9176	-.3742
11	6.0832	-.6913
12	8.2193	-.6795
16	18.8019	-.6564
17	21.9494	-.6534
18	25.296	-.6510
19	28.8415	-.6490
20	32.5855	-.6474

Tabl (2)

K	Re (λ_k)	I_m (λ_k)
1	.6451129	-.3834173
2	3.9999783	-.0208327
3	9.001364	.0142399
4	16.40828	.0083355
5	25.0001322	.052113
6	36.0002513	.0035827
7	49.0004138	.0026129
8	64.0006487	.0020030
9	81.0010011	.0016164
10	100.0014241	.0013056

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3 - The numerical results

The numerical values of λ are obtained using the procedure of section 2. The results are checked to satisfy the equation (7). To express equation (7) in terms of solutions of (1) and (2a, b) let

$$y = C\,Ai(-x-\lambda) + D\,Bi(-x-\lambda) \qquad (14)$$

C and D are constants. Eauation (14) must satisfy (2a, b), it follows that

$$\Delta(\lambda) = Bi(-\lambda)\,[i\sqrt{b}\,Ai(-\lambda-b) + A'i(-\lambda-b)] - Ai(-\lambda)\,[i\sqrt{b}\,Bi(-\lambda-b) + B'i(-\lambda-b)] = 0 \qquad (15)$$

Where Ai and Bi are the Airy functions.

When $|\lambda|$ is large and $|b+\lambda|$ is large using the asymptotic relations of Ai and Bi for complex arguments equation (15) is reduced to

$$\tan\left[\frac{2}{3}(\lambda+b)^{\frac{3}{2}} - \frac{2}{3}\lambda^{\frac{3}{2}}\right] = -i\sqrt{1+\frac{b}{\lambda}} \qquad \arg|\lambda| < \frac{2\pi}{3} \qquad (16)$$

Equation (16) gives the eigenvalues of (1) and (2a, b) when $|\lambda|$ is large.

In order to check the numerical results equation (15) is used with the numerical values of Ai and Bi, for complex argument of any size , are evaluated from [4].

Tabel (1) b = 10

4 - Concluding remarks

The numerical investigation of the self-adjoint boundary value problem has been dealt with extensiely in the past. Since the eigenvalues are mostly real, hence they can be ordered easily, the computation of them becomes straightforward using the prufer transformation and a root finder. But the non-self-adjoint problems are not so. One way of numbering the complex eigenvalues is to correspond them to the eigenvalues of a self-adjoint boundary value problem which can be continued to the non-self-adjoint problem. Of course there may exist other methods of ordering them.

One should remark that the prufer like transformation mentioned in section 2 cans be used without the continuation method for certain other non-self-adjoint boundary value problems. problems with self adjoint boundary conditions, such a,

$$y'' - (2q\cos 2x - \lambda)\,y = 0$$
$$y(0) = 0\ ,\ y(\pi) = 0$$

where q is a ccomplex constant. The eigenvalues of this problem will be the characteristic values of the Mathieue equation associated with odd periodic solutions. An approimation to the eigenvalues of this problem is given in [5] Here $\lambda_k = k^2$ and the problem is self- adjoint when q is real.

Now take for example q= (1 + i) / then λ_k, k = 1, 2, ..., 10 are computed by the method of section2 without the continuation. The results are shown below in a good agreement with those in [5].

Table (2)

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$$(\lambda_n - \lambda_m) \int_0^b y_m y_n \, dx = 0 \quad (6)$$

Now to prove that $Im(\lambda_n) \leq 0$ let $\bar{\lambda}$ denotes the complex conjugate of λ. Since q (x) is real valued function, $y(x, \bar{\lambda}) = \overline{y(x,\lambda)}$ and $y'(x, \bar{\lambda}) = \overline{y'(x,\lambda)}$, rewriting equations (3), (4) and (6) with $\bar{\lambda}_n$ in place of λ_n and taking into consideration the complex conjagate of the boundary conditions we obtain

$$Im(\lambda_n) = -\sqrt{b}\, |y_n(b)|^2 / \int_0^b |y_n(x)|^2 dx \leq 0$$

2. A numerical method

To find the numerical values of the eigenvalues of (1), (2a, b) a combination of prufer-like transformation and a pseudo arclength continuation method [1] is used, the results then checked using the eigenrelation

$$\Delta(\lambda) = 0 \quad (7)$$

let $\frac{y(x,\lambda)}{y'(x,\lambda)} = \tan\theta(x,\lambda)$ then

$$\frac{d\theta}{dx} = \cos^2\theta + (x+\lambda)\sin^2\theta \quad (8)$$

$$\theta(0) = 0 \quad (9)$$

$$\theta(b,\lambda) = \text{Arctan}\left(\frac{-i}{\sqrt{b}}\right)$$

$$= k\pi + \frac{i}{4}\text{Ln}\left(\frac{\sqrt{b}+1}{\sqrt{b}-1}\right)^2 \quad (10)$$

A continuation parameter β is introduced in the following way. Equation (2b) is written as

$$\sin\beta\, y'(b) = i\sqrt{b}\, y(b)\cos\beta \quad (11)$$

when $\beta = 0$ the boundary value problem (1), (2a) and (11) is self-adjoint and its eigenvalues are $\lambda_1 < \lambda_2 < \lambda_3 < .....$ all real and can be computed using any one of the known packages such as the nag routine D02 KDF [6]. When $\beta = \frac{\pi}{4}$ the problem (1), (2a) and (11) is non-self-adjoint.

Starting with λ as a function of β, Euler's method is used as a predictor and Newton's method as a corrector. To improve the efficiency of the method one can use a higher order method for a predictor . Integrating the system

$$\frac{d\lambda}{d\beta} = -i\cos^2\theta / \sqrt{b}\; z\cos^2\beta \quad (12)$$

$$\lambda(0) = \lambda_0$$

Where $z = \frac{\partial\theta}{\partial\lambda}$ and λ_0 can be any one of λ_k's of the self-adjoint problem. z and θ at (b, β) are obtained from the integration of the system

$$\frac{d\theta}{dx} = \cos^2\theta + (x+\lambda)\sin^2\theta$$

$$\frac{dz}{dx} = (x+\lambda-1)\sin 2\theta + \sin^2\theta$$

$$\theta(0) = 0, z(0) = 0.$$

After each integration step of the system (12) λ is corrected using the system

$$\tan\theta + i\tan\beta/\sqrt{b} = 0$$

$$\dot{\lambda}^T(\lambda - \lambda_0) + \dot{\beta}_0(\beta - \beta_0) - s = 0 \quad (13)$$

where dot denotes differentiation with respect to s, and s is the pseudo arclength parameter.

A Numerical Method for the Eigenvalues of Non - self - adjoint Boundary Value Problems

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Abstract

A numerical method is suggested for the computation of eigenvalues of some non-self-adjoint boundary value problem using the prufer transformation, shooting method and a pseudo arclength continuation method. The method is tested on the Mathieue equation the results are in good agreement with the existing ones. Then the method is applied for a non-self-adjoint equation with unknown eigenvalues.

1 - Introduction

The following non-self-adjoint boundary value problems is considered,

$$y'' + (q(x) + \lambda)\, y = 0 \qquad \text{on } [0, b] \tag{1}$$

$$y(0) = 0 \tag{2a}$$

$$y'(b) = i\sqrt{b}\, \beta y(b) \tag{2b}$$

where q (x) = α (x) is a real and smooth function of x, y is a complex valued function of x, α and β are positive constants. Without loss of generality let α =1 then it can be seen easily that β = 1.

Let $y_m(x)$, $y_n(x)$ be eigenfunctions of (1) satisfying (2a, b), λ_m and λ_n are the corresponding eigenvalues then one can prove that, if $\lambda_m \neq \lambda_n$ the eigenfunctions are orthogonal, ie

$$\int_0^b y_m y_n \, dx = 0$$

and that the imaginary part of λ_n, Denoted as Im (λ_n), is less than or equal to zero for all n.

To prove the above statements we write the differential equations satisfied by y_n and y_m and multiplying them by y_m and y_n respectively then

$$y_m[y_n'' + (q + \lambda_n)\, y_n] = 0 \tag{3}$$

$$y_n[y_m'' + (q + \lambda_m)\, y_m] = 0 \tag{4}$$

subtracting (4) from (3) gives

$$y_m y_n'' - y_n y_m'' + (\lambda_n - \lambda_m)\, y_m y_n = 0 \tag{5}$$

Rewriting the first two terms of (5) as a perfect derivative, integrating from 0 to b and using the boundary conditions (2a, b) gives