$$g(z) = \frac{1}{f(z)} = \prod_{i=1}^{n} \frac{1}{(z - z_i)}$$

With n poles at $z_i \in D$, $i = 1, ..., n$. This has the charactristic - n on ∂D.

For the new required function $\hat{m}$ difined by $m = g(z)\hat{m}$ we have the homogeneous boundary-value problem;

$$(5.2) \qquad \frac{\partial \hat{m}}{\partial \bar{z}} = \bar{A}\hat{m} + (\frac{g(\bar{z})}{g(z)}B)\ \bar{\hat{m}}$$

$Re\ ((\bar{P}g)\hat{m}) = 0$ on ∂D of zero characteristic. Here the new coefficient $\bar{B} = \frac{g(\bar{z})}{g(z)}B$ has bounded singularities at the poles z_i, like the corresponding coefficients in *(4.1)* and we conclude, as in section 4, that *(5.2)* has in $\bar{D}$ a Holder-continuous, non-vanishing solution

$$\hat{m}_h = \lambda e^{\hat{m}_h} (\lambda \neq 0)$$
$$m(z, \bar{z}) = \lambda g(z) e^{\hat{m}_h(z, \bar{z})}$$

is a solution of the homogeneous boundary-value problem*(5.1)* of positive charactristic n with poles $z_i \in D$. Then:

Theorem 5.1 *The homogeneous boundary-value problem (5.1) of positive characteristic n has a solution of the form*

$$m(z, \bar{z}) = \lambda(\prod_{i=1}^{n} \frac{1}{(z - z_i)})e^{\hat{m}_h(z,\bar{z})}$$

with arbitrarily assignable poles $z_i \in D\ i = 1, ..., n$.

Then; Each of the non-trivial solutions of homogeneous boundary-value problem with integral characteristics n so far discussed has no zeros on the boundary ∂D, $m \neq 0$. Since the vectors $\bar{m}$ and $\bar{P}$ are perpendicular on ∂D, $\bar{m}$ has the same characteristic as $\bar{P}$. So with each solution m of the differential equation *(5.1)*, which has no zeros on ∂D, we can always associate a homogeneous boundary-value problem of integral characteristic. Hence:

Theorem 5.2 *The characteristic*

$$n = \frac{1}{2\pi} \int_{s=0}^{l} \frac{d}{ds} (arg\ \bar{m})\ ds$$

on ∂D, (5.3), of a solution m of the homogenous differential equation (5.1) which has neither zeros nor poles on ∂D is equal to the difference between the numbers of its poles and zeros in D

$$n = [(number\ of\ poles) - (number\ of\ zeros)]$$

where multiple zeros or poles are to be counted according to their multiplicity.

Every such solution m can be expressed in terms of a meromorphic function of z only, in D:

$$m(z, \bar{z}) = \lambda f(z) e^{\hat{m}_h(z, \bar{z})}$$

and the poles and zeros of f(z) coincide with those of m.

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[4] *[I.N]:I. N. VEKUA, Generalized analytics (Pergamon press 1962).*

$$m_{2k-1}(z_j , \bar{z}_j) = i\delta_{jk}$$

This construction can be carried out for every integer $k = 1, ..., n$. Hence we have:

Theorem 4.1 *Theorem 1:*

For n distinct points $z_j \in D$ $j = 1,..., n$, and for $2n+1$ linearly independent continuous solutions $\tilde{m}_\gamma$, $\gamma = 0, 1, ..., 2n$, of (4.1), it is always possible to choose $2n + 1$ continuous solutions m_γ as linear combinations of the $\tilde{m}_\gamma$ with real coefficients λ_γ such that the m_γ satisfy the equations:

$$\begin{aligned} m_0(z_j , \bar{z}_j) &= 0 \\ m_{2k}(z_j , \bar{z}_j) &= \delta_{jk} \\ m_{2k-1}(z_j , \bar{z}_j) &= i\delta_{jk} \end{aligned} \qquad (4.5)$$

at n prescribed points $z_j \in D$.

Theorem (4.1) implies that there are at most 2n+1 linearly independent continuous solutions of (4.1) ([I.N],p.*250*). That is to say, if m is a continuous solution of (4.1), then m can be expressed as

$$m = \sum_{\gamma=0}^{2n} \lambda_\gamma m_\gamma$$

where the cofficients λ_γ are uniquely determined. To see this we form $m(z_j , \bar{z}_j)$ $j = 1, ..., n$ and obtain from (4.5)

$$\begin{aligned} \lambda_{2j} &= Rem(z_j , \bar{z}_j) \\ \lambda_{2j-1} &= Imm(z_j , \bar{z}_j) \end{aligned}$$

The function

$$m - \sum_{\gamma=1}^{2n} \lambda_\gamma m_\gamma$$

is then a solution of (4.1) and it vanishes at z_j, $j=1, ..., n,(4.5)$. Therefore, by corollary 1 from Theorem (3.3) there is a unique, real constant λ_0 such that

$$m - \sum_{\gamma=1}^{2n} \lambda_\gamma m_\gamma = \lambda_0 m_0$$

in $\bar{D}$ then

$$m = \sum_{\gamma=0}^{2n} \lambda_\gamma m_\gamma$$

We now ask whether there are exactly $2n + 1$ linearly independent solutions of (4.1). The answer suplied by:

Theorem 4.2 *For n distinct chosen points $z_1, ..., z_n \in D$ it is possible to determine $2n + 1$ non-trivial continuous solutions $m_0, ..., m_{2n}$ of (4.1) such that the equations:*

$$\begin{aligned} m_0(z_j , \bar{z}_j) &= 0 \\ m_{2k}(z_j , \bar{z}_j) &= \delta_{jk} \\ m_{2k-1}(z_j , \bar{z}_j) &= i\delta_{jk} \\ k,j &= 1,...,n \end{aligned} \qquad (4.5)$$

hold at these points. These $2n + 1$ functions from a basis of the $2n + 1$ dimensional function space of the solutions of the boundary-value problem (4.1) (proof in [W.W]).

5 - Solutions of Boundary-value problems of positive characteristic with poles

We shall try to represent the solutions of the homogeneous boundary-value problem of positive charactristic:

$$\frac{\partial m}{\partial \bar{z}} = \bar{A}m + B\bar{m} \qquad (5.1)$$

$Re\ (\bar{P}m) = 0$ on ∂D;

$$n = \frac{1}{2\pi} \int_{s=0}^{l} \frac{d}{ds} (arg\ \bar{P})\ ds$$

$|P| \neq 0$.

In a simple way, similar to that of for negative characteristics. To this end we consider the complex function of z only:

(4.2) $$m_0 = \sum_{\gamma=1}^{2n} \lambda_\gamma^{(0)} \tilde{m}_\gamma = \sum_{\gamma=1}^{2n} \lambda_\gamma^{(0)} (\tilde{u}_\gamma + i\tilde{v}_\gamma)$$

shall have zeros at the chosen points $m_0(z_i, \bar{z}_i) = 0$, $i = 1, \ldots, n$. This yields $2n$ linear equations for the $2n + 1$ real coefficients $\lambda_\gamma^{(0)}$:

$$\sum_{\gamma=1}^{2n} \lambda_\gamma^{(0)} \tilde{u}_\gamma (z_i, \bar{z}_i) = 0$$

$$\sum_{\gamma=1}^{2n} \lambda_\gamma^{(0)} \tilde{u}_\gamma (z_i, \bar{z}_i) = 0$$

$i = 1, \ldots, n$. They have at least one solution $\lambda_\gamma^{(0)}$ in which not all the $\lambda_\gamma^{(0)}$ vanish. Hence by re-numbering the functions $\tilde{m}_\gamma$ if necessary, we can always ensure that the condition $\lambda_0^{(0)} \neq 0$ holds. The functions m_o, $\tilde{m}_1, \ldots, \tilde{m}_{2n}$ are then linearly independent, for otherwise, by (4.2), their linear dependence would imply that of $\tilde{m}_0, \tilde{m}_1, \ldots, \tilde{m}_{2n}$. Choosing some fixd integer k. $1 \leq k \leq n$, we now try to choose two linear combination of $\tilde{m}_1, \ldots, \tilde{m}_{2n}$, say:

(4.3)
$$m_{2k} = \sum_{\gamma=1}^{2n} \lambda_\gamma^{(2k)} \tilde{m}_\gamma$$

$$m_{2k-1} = \sum_{\gamma=1}^{2n} \lambda_\gamma^{(2k-1)} \tilde{m}_\gamma$$

With real coefficients $\lambda_\gamma^{(2k)}$, $\lambda_\gamma^{(2k-1)}$ which will have $n - 1$ common zeros $z_j \neq z_k$.

There is certainly no non-trivial linear combination with zeros at all the $z_1, \ldots, z_n$ and therefore also at z_k, for if there were, then by corollary 1 from theorem (3.3), the m_0, $\tilde{m}_1, \ldots, \tilde{m}_{2n}$ would be linearly dependent.

Although m_{2k} and m_{2k-1} cannot be zero at z_k, we might at least seek to prescribe particular values for these functions at z_k. If we *demand* that $(z_k, \bar{z}_k) = 1$, then we have $2n$ linear equations:

$$\sum_{\gamma=1}^{2n} \lambda_\gamma^{(2k)} \tilde{u}_\gamma (z_j, \bar{z}_j) = \delta_{jk}$$

(4.4)
$$\sum_{\gamma=1}^{2n} \lambda_\gamma^{(2k)} \tilde{\gamma}_\gamma (z_j, \bar{z}_j) = 0$$

for the $2n$ real, unknown coefficients $\lambda_\gamma^{(2k)}$.

To investigate the determinant of coefficients of this linear system, we consider the linear independence of $m_0, m_1, \ldots, \tilde{m}_{2n}$. The equation

$$\sum_{\gamma=1}^{2k} x_\gamma \tilde{m}_\gamma = x_0 m_0$$

immediately inplies that all the χ_γ vanish. Consequently there are no real numbers χ_γ such that

$$\sum_{\gamma=1}^{2k} (x_\gamma)^2 > 0$$

and such that the relation

$$\sum_{\gamma=1}^{2n} x_\gamma \tilde{m}_\gamma = (z_j, \bar{z}_j) = 0$$

$j = 1, \ldots n$ could hold. Therefore the determinant of coefficients of the system (4.4) dose not vanish.

Accordingly, the linear system(4.4) has precisely one solution $\lambda_\gamma^{(2k)}$ with which the function m_{2k} given by (4.3) and (4.4) has the property that

$$m_{2k}(z_j, \bar{z}_j) = \delta_{jk}$$

and satisfies (4.1), $j = 1, \ldots, n$. In the same way we can determine the real linear factors $\lambda_\gamma^{(2k-1)}$ as the solution of

$$\sum_{\gamma=1}^{(2k-1)} \lambda_\gamma^{(2k-1)} \tilde{u}_\gamma (z_j, \bar{z}_j) = 0$$

$$\sum_{\gamma=1}^{2n} \lambda_\gamma^{(2k-1)} \tilde{v}_\gamma (z_j, \bar{z}_j) = \delta_{jk}$$

and hence obtain the solution m_{2k-1} of (4.1) with the property

$$v = \frac{X}{p^2 + q^2}(p\hat{u}_H - q\hat{v}_H) = Xv_H$$

Where, in D

$$u^2_H + v^2_H \geq 0$$

and the real constant X can be chosen arbitrarily

By principal theorem (3.1), there is in $\bar{D}$ a Holder-continuous, non-vanishing solution of the boundary-value problem and therefore also of (3.3), provided that the discontinuities of the coefficients of (3.3) are bounded. We denote this homogeneous solution of (3.3) by $\hat{m}_h$. The function $m_h = f(z)\hat{m}_h$ obtained in this way satisfies the differential equation (3.1) everywhere in D, since m_h is continuous. We collect these results together the:

Theorem 3.3: *The homogeneous boundary-value problem of negative characteristic $n < 0$ has the family of solution:*

$$m_h = \lambda f(z) e^{\hat{m}_h(z, \bar{z})}$$

where λ is an arbitrary real constant and $f(z)$ is an entire rational function of z only, having n arbitrarily assignable zeros,

$$z_\gamma = x_\gamma + iy_\gamma \qquad \gamma = 1,2,\ldots,n$$

The solution therefore contains $(2n + 1)$ arbitrarily real constant λ, $x_1, y_1, \ldots, x_n, y_n$

From this theorem we can deduce directly the following two important corollaries:

1) If two continuous solutions m_1 and m_2 of the same homogeneous boundary-value problem of characteristic $n<0$ have the same zeros $z_1, \ldots, z_n$, *then there are two real numbers* χ_1, χ_2 such that $\chi_1 m_1 = \chi_2 m_2$ holds throughout $\bar{D}$.

2) A continous solution m of the homogeneous boundary-value problem of characteristic $n < 0$ which has $n + 1$ distinct zeros in $\bar{D}$, With n of them lying in D, must vanish identically in $\bar{D}$. To solve the inhomogeneous boundary-value problem with $\phi \neq 0$, we again introduce a new required function $\hat{m}$ by means of (3.2). $\hat{m}$ now has to be a solution of the differential equation (3.3) with the inhomogeneous boundary-condition of characteristic zero $Re(\bar{P}\hat{m}_p) = \phi(s)$ on ∂D, which, we can reduce to the corresponding inhomogeneous boundary-value problem. Using the function $\hat{m}_h$ introduced above for $f(z)$ and the homogeneous problem, we now obtain, as the general solution m of the new problem, the representation

$$m = f(z)[\lambda e^{\hat{m}_h(z, \bar{z})} + \hat{m}_p]$$

which likewise, has $2n + 1$ arbitrary real constants $\lambda, x_1, y_1, \ldots, x_n, y_n$ and has the same zeros as $f(z)$.

4 - The solution set for the homogeneous problem of negative characteristic

If m_1, m_2 are distinct solutions of the homogeneous problem of negative characteristic n:

$$(4.1) \qquad \frac{\partial m}{\partial \bar{z}} = \bar{A}_m + B\bar{m}$$

$Re(\bar{\gamma} m) = 0$ on $\partial D \mid \bar{\gamma} \mid > 0$ then any linear combination of them $m = \lambda_1 m_1 + \lambda_2 m_2$ *with real* λ_1, λ_2 is also a solution of (4.1). The general solution of (4.1) contains, according to the Theorem(3.3), $2n + 1$ arbitrary real constants. It may therefore be conjectured that there are $2n + 1$ linearly independent solutions of (4.1).

Suppose, first, that we already know $2n + 1$ linearly independent solutions $\tilde{m}_0, \ldots, \tilde{m}_{2n}$ of (4.1), no pair of these solutions can have the same zeros, since, if the zeros are prescribed, only one constant is available. But we might try to obtain at least some coincidence of the zeros by a linear combination of the $\tilde{m}_\gamma$. To do this, we first choose n distinct points $z_i \in D$ which will then be kept fixed throughout discussion. We then demand that the linear combination:

also hold at points of $t_1 \cap$ D.

Therefore $f(z)$ is, in D, a regular, analytic function of z only. As we know, it can have only a finite number of zeros in D. Hence, it follows by (1.4) that the solution m of the system (1.4) can have only isolated zeros in D.

3- Boundary - Value Problem of Negative Characteristic

We now investigate the general solution m of the problem;

$$\frac{\partial m}{\partial \bar{z}} = \bar{A}m + B\bar{m} \qquad (3.1)$$

$$Re(\bar{\gamma}m) = \phi(s)$$

on ∂D. With a boundary vector-family $\gamma = \alpha(s)+i\beta(s)$, $|\gamma| \neq 0$ of negative characteristic:

$$n = \frac{1}{2\pi}\int_{s=0}^{l} \frac{d}{ds}\left(\arctan\frac{\alpha(s)}{\beta(s)}\right) d.s$$

We deal first with the homogeneous problem $\phi=0$, since, the function of a single variable z:

$$f(z) = \prod_{i=1}^{n}(z - z_i)$$

$z_i \in D$. has the characteristic + n on ∂D.

We now introduce a new required function $\hat{m}(z\,.\,\bar{z})$ by putting

$$m(z,\bar{z}) = f(z)\hat{m}(z\,,\bar{z}) \qquad (3.2)$$

and thus find that $\hat{m}$ must satisfy the boundary condition

$$Re(\bar{\gamma}m) = Re(\bar{P}\hat{m}) = 0$$

on ∂D, that $\bar{P} = \bar{\gamma}f$ which has the characteristic zero.

We already have met the new boundary - condition for $\hat{m}$ on substituting (3.2) in to the differential equation (3.1) we obtain, since $\frac{\partial f}{\partial \bar{z}} = 0$, and so the differential equation for $\hat{m}$ is:

$$\frac{\partial \hat{m}}{\partial \bar{z}} = \bar{A}\hat{m} + \left(\frac{\overline{f(z)}}{f(z)}B\right)\bar{\hat{m}} \qquad (3.3)$$

Thus $\hat{m}$ is to be the solution of a homogeneous boundary - value problem of characteristic zero for the differential equation (3.3). However, in (3.3) the coefficient

$$\bar{B} = \frac{\bar{f}}{f}B$$

$\bar{B}$ therefore has bounded discontinuities at the n zeros of f.

By Principal theorems:

Theorem 3.1 *If the coefficients $a^i, b^i, \tilde{a}^i, \tilde{b}^i$ are C^{2+a}-functions and $c, \tilde{c}, e, \tilde{e}$, are C^{1+a}-functions and f, $\tilde{f}$ are C^{0+a}- functions in the closure $\bar{D}$ of a bounded, simply connected, and sufficiently small domain D in the (x,y)- plane having a boundary ∂D with holder- continuous curvature, then the system:*

$$a^1u_x + a^2u_y + b^1v_x + b^2v_y = cu + ev + f$$

$$\tilde{a}^1u_x + \tilde{a}^2u_y + \tilde{b}^1v_x + \tilde{b}^2v_y = \tilde{c}u + \tilde{e}v + \tilde{f}$$

If it is elliptic in $\bar{D}$, has always a solution-pair u, v such that the non-trivial solutions u_h, v_h of the problem for the homogeneous differential equations ($f = \tilde{f} = 0$ with $u = 0$ on ∂D satisfy the condition $u_h^2 + v_h^2 \neq 0$ in $\bar{D}$.

If u_h, v_h is one non-trivial solution of the homogenous problem, then χu_h, χv_h, where χ is constant, form the whole class of solutions. further, the inhomogeneous problem with $u = \phi(s)$ on ∂D and f, $\tilde{f} \neq 0$ always has a family of solutions, which can be made unique by any choice of a boundary norm or suitable surface norm.

Theorem 3.2 *The general, linear, homogeneous, boundary-value problem with zero characteristic has, in a bounded, simply-connected domain D with a boundary ∂D having continuous curvature, if the coefficients a, b, $\tilde{a}$, $\tilde{b} \in C^{1+a}$ in $\bar{D}$ and if p, q$\in C^{1+a}$ ($\alpha \geq 0$) in $\bar{D}$, a set of solutions*

$$u = \frac{X}{p^2 + q^2}(p\hat{u}_H - q\hat{v}_H) = Xu_H$$

is open and therefore measurable.

The system (2.2) corresponds to the normal form (1.1) with $\alpha = b = \tilde{\alpha} = \tilde{b} = 0$. Therefore the representation formula, ([W.W]):

$$u(\xi, \eta) = \iint_D [-\Im dy + \tilde{\Im} dx, dG^1] - \oint_{\partial D} u d_n G^1$$

(2.3)

$$v(\xi, \eta) = -\iint_D [\Im dx + \tilde{\Im} dy, d\bar{G}^{11}] + \oint_{\partial D} u d\bar{G}^{11} + \frac{1}{\iint \tau^2 [dx, dy]} \iint_D v\tau^2 [dx, dy]$$

Can be applied, with:

$$\Im = -2(a + \alpha + H \cos q)$$

$$\tilde{\Im} = -2(-b + \beta + H \sin q)$$

this gives:

$$\tilde{v}(\xi, \eta) = 2 \iint_D (-a + \alpha + H \cos q) G_x^1 - (-b + \beta + H \sin q) G_y^1 [dx, dy]$$

(2.3a)

$$- \oint_{\delta D} \tilde{u} d_n G^1$$

$$v(\xi, \eta) = 2 \iint_D (a + \alpha + H \cos q) G_y^{11} - (-b + \beta + H \sin q) G_x^{11} [dx, dy]$$

(2.3b)

$$+ [\oint_{\partial D} \sigma ds]^{-1} \oint_{\partial D} \tilde{r} \sigma ds + \oint_{\partial D} \tilde{r} dG^{11}$$

Have been arbitrarily prescribed, *(2.3.a.b)* are the solution formula for (2.2). We have still to show that the function f, Holder–Continuous in $\bar{D}$, which is given by: $f = me^{-\tilde{m}}$ is in D, a regular, analytic function of z only, but, by construction, f already satisfies the Cauchy-Riemann system of differential equations: $\frac{\partial f}{\partial \bar{z}} = 0$ at all points $(z, \bar{z}) \in$ D-t. Hence, by the theorem on removable singularities *([A.R], p.315)*, it follows that f also satisfies these differential equations at the isolated zeros of m. At the remaining zeros $(z_0, \bar{z}_0) \in t_2 - t_1$ of m we find, by virtue of the continuity of $\tilde{m}$ and the system (1.4), and using the mean-value theorem for functions of two real variables,

$$\lim_{z \to z_0} \frac{f(z, \bar{z}) - f(z_0 - \bar{z}_0)}{z - z_0} = e^{-\tilde{m}(z_0, \bar{z}_0)} \frac{\partial m}{\partial z} (z_0, \bar{z}_0)$$

So this limit exist at z_0 and is independent of the directio of approach to z_0. Consequently we have at $(z_0, \bar{z}_0)$

$$\lim_{\delta \to 0} \frac{f(z_o + \delta)}{\delta} = f_x = \lim_{\delta \to 0} \frac{f(z_0 + i\delta)}{i\delta} = -if_y$$

or

$$(f_x + if_y) = \frac{\partial f}{\partial \bar{z}} = 0$$

$$\hat{B} = B\frac{\overline{P}}{P} = Be^{-2i\theta}$$

$$\hat{C} = \overline{P}C$$

Then, new boundary–condition for $\hat{m}$ is:
$Re(\hat{m}) = f(s)$.

2- The Behaviour of solution of a Elliptic system at their Zeros

We assume that in $\overline{D}$ the coefficients A,B,C are bounded and that, at most, they are discontinuous only at a finite number of points $(z_j, \overline{z}_j) \in \overline{D}$, $j = 1,2,...,k$. We shall denote the set of these points by:

$$t_1 = \bigcup_{j=1}^{K} (z_j, \overline{z}_j)$$

At the zeros of m, the system (1.4) will be reduced to the Cauchy- Rimann differential equations at these points. We can therefore expect the behaviour of the solution m to be similar to that of a regular, analytic function in some neighbourhood of the zero. The following theorem holds:

Theorem 2.1 *Let $m(z, \overline{z})$ be a solution of the (1.4) which is continuously differentiable in $D - t_1$ and Holder - continuous in $\overline{D}$. Let the coefficients A, B, C be Holder-continuous in $\overline{D} - t_1$. Let the boundary ∂D have no double-points and have Holder -continuous tangents. Then there exists, corresponding to m, a function $\tilde{m}(z, \overline{z})$ Holder -continuous in $\overline{D}$ and such that:*

$$(2.1) \qquad f(z) = m(z, \overline{z})e^{-\tilde{m}(z, \overline{z})}$$

Is a regular, analytic function in D of z only. Moreover, the imaginary part or the real part of $\tilde{m}$ be prescribed aritrarily, so long as $\tilde{m}$ is kept Holder -continuous.

Proof.

We use the fact that $f(z)$ is a regular, analytic function provided that the Cauchy-Riemann differential equations:

$$(2.3) \qquad \frac{\partial f}{\partial \overline{z}} = 0$$

hold throughout D. In that case, we obtain for $\tilde{m}$ from (1.4):

$$\frac{\partial f}{\partial \overline{z}} = e^{-\tilde{m}}(\overline{A}m + B\overline{m} + \overline{C} - m\frac{\partial \tilde{m}}{\partial \overline{z}}) = 0$$

the differential- equation system:

$$\frac{\partial \tilde{m}}{\partial \overline{z}} = \overline{A} + \frac{\overline{m}}{m}B + \frac{C}{m}$$

throughout $D - t_1$, when $m \neq 0$.
We therefor let

$$t_2 = \{(z, \overline{z}) \in D : m(z, \overline{z}) = 0\}$$

denote the point-set of zeros of m, and put

$$t = t_1 \cup t_2$$

Writing:

$$B = H$$

$$\frac{B}{H}\frac{\overline{m}}{m} = e^{iq}$$

$$A = a + ib$$

$$\frac{C}{m} = \alpha + i\beta$$

$$\tilde{m} = \tilde{u} + i\tilde{v}$$

And using (1.2), we can write the above system as the real system:

$$\tilde{u}_x - \tilde{v}_y = 2(a + \alpha + H\cos q)$$

$$(2.2) \qquad \tilde{u}_y + \tilde{v}_x = 2(-b + \beta + H\sin q)$$

In D - t the right - hand members of (2.2) are given Holder-continuous functions, in t we put them equal to zero. They are integrable in $\overline{D}$, and they are bounded, measurable functions, since D-t

γ has an important influence on the character of the solutions of (1.1). We therefore define the concept of the characteristic of a periodic vector - family P on ∂D.

The vectors P (s) are anchored at the origin of the complex plane. The end- point of P(s) describes, for $0 \leq s \leq 1$, a curve in the complex plane.

The number n of complete revolutions made by P(s) when ∂D is traced out in the positive direction is called characteristic of P$= p + iq$:

$$(1.2) \qquad n = \frac{1}{2\pi} \int_{s=0}^{l} \frac{d}{ds} (\arctan(\frac{q}{p}))\, d.s$$

If P rotates in the positive sense, then $n > 0$ otherwise, $n < 0$. We make the further difinition:

The boundary-value problem for (1.1) has the characteristic n if the boundary family $\bar{\gamma} = e^{-i\phi}$ has the characteristic n.

We introduce complex - valued functions of two isotropic parameters:

$$z = x + iy$$
$$\bar{z} = x - iy$$

then for a perfect differential invariant under a transformation of coordinates, of a function $\phi(x,y) = \phi(z, \bar{z})$ we have:

$$d\phi = \phi_z dz + \phi_{\bar{z}} d\bar{z} = \phi_x dx + \phi_y dy = (\phi_z + \phi_{\bar{z}})\, dx + i\,(\phi_z - \phi_{\bar{z}}) dy$$

And hence the relation:

$$(1.3) \qquad \phi_x = \phi_z + \phi_{\bar{z}}, \qquad \phi_y = i(\phi_z - \phi_{\bar{z}})$$

Substituting these into the system of differential equations (1,1), we obtain;

$$u_z + u_{\bar{z}} - i(v_z - v_{\bar{z}}) = au + bv + c$$
$$i(u_z - u_{\bar{z}}) + v_z + v_{\bar{z}} = \tilde{a} + \tilde{b}v + \tilde{c}$$

And hence we obtain;

$$2(u_{\bar{z}} + iv_{\bar{z}}) = (a + i\tilde{a})u + (b + i\tilde{b})v + (c + i\tilde{c}) \quad \text{in } D$$
$$2(u_z - iv_z) = (a - i\tilde{a})u + (b - i\tilde{b})v + (c - i\tilde{c})$$

Since u,v are real functions, it follows, using (1.3) that the 2nd equation is the complex conjugate of the lst equation. Introducing the complex-valued function:

$$m(z, \bar{z}) = u + iv$$
$$A(z, \bar{z}) = \frac{1}{4}[a + \tilde{b} + i(b - \tilde{a})]$$
$$B(z, \bar{z}) = \frac{1}{4}[a - \tilde{b} + i(\tilde{a} + b)]$$
$$C(z, \bar{z}) = \frac{1}{2}(c + i\tilde{c})$$

We find, taking the difinitions (1.3) into account the system (1.1) is equivalent to the single complex equation:

$$(1.4) \qquad \frac{\partial m}{\partial \bar{z}} = \bar{A}m + B\bar{m} + C$$

since the complex conjugate of (1.4) is:

$$\frac{\partial \bar{m}}{\partial z} = A\bar{m} + \bar{B}m + C$$

If on the other hand, z and $\bar{z}$ are regarded as independent variables, then the system (1.1) corresponds to the two equations (1.4),(1.5).

Let m is to be a solution of (1.4), let the boundary vector-family be:

$$P(s) = p(s) + iq(s)$$

and its continuatin on to $\bar{D}$ be $P(z, \bar{z}) = R(z, \bar{z})\, e^{i\theta(z, \bar{z})}$ with $|P| = R \neq 0$ in $\bar{D}$.

The boundary condition reads:

$Re(\bar{P}m) = pu + qv = f(s)$. Writing:

$\hat{m}(z, \bar{z}) = \bar{P}(z, \bar{z}) m(z, \bar{z})$, we obtain for $\hat{m}$ from (1.4) the equation:

$$(1.5) \qquad \frac{\partial \hat{m}}{\partial \bar{z}} = \bar{\hat{A}}\hat{m} + \hat{B}\bar{\hat{m}} + \hat{C}$$

so that:

$$\hat{A} = A + \frac{\partial P}{\partial z} \frac{1}{P}$$

On a Generalized Analytic Hilbert Type Boundary Value Problem

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Abstract

A linear system of partial differential equations containig two unknowns, and, generalised analytic functions with linear boundary condition is considered. existence and the properties of solution of the boundary value problem relative to characteristic number is studied.

1-Introduction

It is well-known that many problems of mathematical physics may be described by the systems of partial differential equations of first order. One of the basic problems is the Hilbert problem. We investigate on linear partial differential equations that obtain from Hilbert problem (in the [M. A] discussion is about nonlinear form).

In this paper we will consider only systems in the Hilbert normal form:

$$(1.1) \quad \begin{cases} u_x - v_y = au + bv + c = \Im \\ u_y - v_x = \tilde{a}u + \tilde{b}v + \tilde{c} = \Im \end{cases} \quad \text{in D}$$

To begin with; let the coefficients a,b, c, ã, $\tilde{b}$, $\tilde{c}$ be continuously differentiable in $\bar{D}$, Where D is open, bounded, simply-connected set in the complex plane, with piecewise continuous rectifiable boundary ∂D, where $\bar{D} = \partial D \cup D$.

General form of the Boundary-value Problems: Suppose a solution of system (1,1) is required to satisfy the following B. C.:

$$\alpha(s)u(s) + \beta(s)v(s) = f(s)$$

on the ∂D.

We shall assume that $\alpha^2 + \beta^2 \neq 0$, and therefore we may noramalize so that: $\alpha^2 + \beta^2 = 1$. If we regard a solution of (1,1) as a vector (u,v), then the boundary-condition $\alpha u + \beta v = 0$ *on* ∂D implies that the vector (u,v) shall be perpendicular to the vector (α , β) in the complex plane , and so we may also write:

$$\begin{cases} m = u + iv = me^{i\theta} \\ \gamma = \alpha + i\beta = e^{i\phi} \end{cases}$$

$$\alpha u + \beta v = Re\,(\bar{\gamma} m) = Re(e^{-i\phi}m) = f(s)$$

The linear boundary condition can now be interpreted as follows:

1) Re $(\bar{\gamma}m) = 0$ means m on ∂D shall be perpendicular to γ.

2) Re $(\bar{\gamma}m) = f(s)$ means that on ∂D the projection of m on to γ is prescribed.

If we consider, as a particular case of the system (1.1), the Cauchy-Rieman differential equations ($\alpha = b = c = \tilde{\alpha} = \tilde{b} = \tilde{c} = 0$), then the boundary condition: Re $(\bar{\gamma}m) = 0$ for a polynomial $m = \Pi^n_{i=1} (z - z_i)$ with zeros $z_i \in D$ shows that γ on ∂D must execute as many revolution as the number of the zeros z_i, lying in D. when γ is prescribed, therefore, the rotation of