$$|A| \geq 0 \qquad \text{(non-negative matrix).} \qquad (53)$$

it results:

$$0 \leq |A| \leq 1 \qquad (54)$$

In words, the determinant of the autocorrelation matrix is always bounded between zero and unity.

Conclusions:

The autocorrelation function theorem has shown that the autocorrelation coefficients maybe represented as a non-negative definite matrix of Toeplitz form.

The properties of this matrix and its submatrices are:

(i) Real, non-negative eigenvalues.

(ii) $0 \leq |A| \leq 1$; ie, the determinant is always bounded between zero and unity.

It has also been shown that the eigenvalues of the infinite autocorrelation matrix are proportional to the spectral densities at their corresponding frequencies and vice versa.

References:

1. Priestly, M.B., "Spectral Analysis and Time Series", Vol. 1, Academic press, 1982.
2. Bendat, J.S. and Piersol, A.G., "Random Data, Analysis and Measurement Procedures", Wiley-Interscience, 1986.
3. Faghih,N. "State Space Modelling of Time series", International Congress of Mathematicians, Kyoto, Japan, 1991.
4. Champeney, D.C., "Fourier Transforms in Physics", Wiley-Eastern, 1989.
5. Soderstrom, T. and Stoica, P., "System Identification", Prentice Hall, 1989.
6. Widom, H., In "studies In Real and Complex Analysis", (I,I Hirshman, Jr., Ed.), The Mathematical Association of America, Vol 3., pp. 179-209, 1965.
7. Parlett, B.N., "The Symmetric Eigenvalue problem", Prentice-Hall, 1980.
8. Grenander, U. and Szego, G., "Toeplitz forms and their applictions", University of colifornia Press, 1958.
9. Graybill, F.A., "Matrices with Applications in Statistics", Wa ndsworth publishers, 1983.
10. Golub, G.H. and Loan, C.V., "Advanced Matrix computations", Johns Hopkins University press, 1984.

$$\left.\begin{array}{l}\alpha_r T_r C_m = k\, C_m \\ \alpha_r T_r S_m = k\, S_m\end{array}\right\} \text{if } r=m \} = 0 \ \text{if } r \neq m \qquad (39)$$

Then C_m and S_m are eigenvectors of matrix T_r. Using equations (28) and (39) one can write:

$$A = \Delta\omega \sum_{r=0}^{\infty} \alpha_r E(r\Delta\omega)\, T_r \qquad (40)$$

Hence:

$$A C_m = \Delta\omega k \,.\, E(m\Delta\omega)\, C_m \qquad (41)$$

Also:

$$A S_m = \Delta\omega k \,.\, E(m\Delta\omega)\, S_m \qquad (42)$$

α_r has disappeared because $\alpha_r=1$, $r \neq 0$; $\alpha_r=1/2$, $r=0$; in which case twice the constant or zero appears. Therefore, it is evident that C_m and S_m are eigenvectors of matrix A:

$$C_m\, (m = 0 \rightarrow \infty)\,;\, \lambda = \Delta\omega\, E(m\Delta\omega)\,.\,k \qquad (43)$$

$$S_m\, (m = 0 \rightarrow \infty)\,;\, \lambda = \Delta\omega\, E(m\Delta\omega)\,.\,k \qquad (44)$$

And the eigenvalues are represented by $\Delta\omega\,.\,E(m\Delta\omega)\,.\,k$; where K is a non-negative constant.

Therefore the eigenvalues of the autocorrelation matrix are proportional to $E(m\Delta\omega)$ and vice versa.

As A is non-negative definite, then the eigenvalues of A must be real and non-negative, a necessary and sufficient condition for a non-negative definite matrix, [7], [9].

As $\Delta\omega$ and K are also non-negative, then $E(m\Delta\omega)$ will be non-negative. Therefore as the infinite autocorrelation matrix is non-negative definite, then the true power spectrum must be non-negative.

Considering equations (41) and (42), it can be seen that for the infinite autocorrelation matrix, the eigenvalues must appear in pairs, with each pair having associated eigenvectors of the form C_m and S_m. These pairs will be equal with the exception of the case for which m is equal to zero.

4. Determinant Of The Autocorrelation Matrix

It has been shown that the autocorrelation matrix is non-negative definite and of Toeplitz form. The eigenvalues must all be real and non-negative, ie:

$$\lambda_i \geq 0 \qquad (45)$$

The determinant of any square matrix A, is equal to the product of the eigenvalues, [8]. Therefore for a matrix of order N,

$$\prod_{i=1}^{N} \lambda_i = |A| \qquad (46)$$

The trace of a matrix is the sum of its eigenvalues, which is equal to the sum of the diagonal elements, [10]. Hence, for the autocorrelation matrix, from equation (19):

$$\sum_{i=1}^{N} \lambda_i = N R(0) \qquad (47)$$

As the geometric mean is never greater than the arithmetic mean, ie:

$$(a_1 \,.\, a_2 \,.\, \ldots \,.\, a_n)^{\frac{1}{N}} \leq \frac{1}{N}(a_1 + a_2 + \ldots + a_n), \qquad (48)$$

Then for the autocorrelation matrix, by substituting from equations (46) and (47) into (48):

$$|A|^{\frac{1}{N}} \leq \frac{N R(0)}{N} \qquad (49)$$

or

$$|A|^{\frac{1}{N}} \leq R(0) \qquad (50)$$

As it is seen from equations (17), (18) and (19), for the autocorrelation:

$$R(0) = 1 \qquad (51)$$

Therefore, it yields:

$$|A| \leq 1 \qquad (52)$$

and since,

Thus, T_r maybe represented by:

$$T_r = \begin{bmatrix} \cos 0 & 0 & 0 & 0 & \cdots \\ 0 & \cos \phi_r & 0 & 0 & \\ 0 & 0 & \cos 2\phi_r & 0 & \\ 0 & 0 & 0 & \cos 3\phi_r & \\ , & , & , & , & \ddots \end{bmatrix} \begin{bmatrix} \cos 0 & \cos \phi_r & \cos 2\phi_r & \cdots \\ \cos 0 & \cos \phi_r & \cos 2\phi_r & \cdots \\ \cos 0 & \cos \phi_r & \cos 2\phi_r & \cdots \\ \cos 0 & \cos \phi_r & \cos 2\phi_r & \cdots \\ , & , & , & \cdots \end{bmatrix} \quad (31)$$

$$+ \begin{bmatrix} \sin 0 & 0 & 0 & 0 & \cdots \\ 0 & \sin \phi_r & 0 & 0 & \\ 0 & 0 & \sin 2\phi_r & 0 & \\ 0 & 0 & 0 & \sin 3\phi_r & \\ , & , & , & , & \ddots \end{bmatrix} \begin{bmatrix} \sin 0 & \sin \phi_r & \sin 2\phi_r & \cdots \\ \sin 0 & \sin \phi_r & \sin 2\phi_r & \cdots \\ \sin 0 & \sin \phi_r & \sin 2\phi_r & \cdots \\ \sin 0 & \sin \phi_r & \sin 2\phi_r & \cdots \\ , & , & , & \cdots \end{bmatrix}$$

Furthermore, introducing vectors C_m and S_m, as:

$$C_m = \begin{bmatrix} \cos 0 \\ \cos \phi_m \\ \cos 2\phi_m \\ \cos 3\phi_m \\ , \\ , \end{bmatrix} \qquad S_m = \begin{bmatrix} \sin 0 \\ \sin \phi_m \\ \sin 2\phi_m \\ \sin 3\phi_m \\ , \\ , \end{bmatrix} \quad (32)$$

where,

$$\phi_m = m \Delta\omega\Delta\tau \quad (33)$$

Then T_r maybe expressed as:

$$T_r = \begin{bmatrix} \cos 0 & & & \\ & \cos \phi_r & & \\ & & \cos 2\phi_r & \\ & & & \ddots \end{bmatrix} \begin{bmatrix} -C_r^t- \\ -C_r^t- \\ -C_r^t- \end{bmatrix} + \begin{bmatrix} \sin 0 & & & \\ & \sin 2\phi_r & & \\ & & \sin 2\phi_r & \\ & & & \ddots \end{bmatrix} \begin{bmatrix} -S_r^t- \\ -S_r^t- \\ -S_r^t- \end{bmatrix} \quad (34)$$

$-C_r^t-$ and $-S_r^t-$ denote row vectors, ie transposes of C_r and S_r respectively.

Consider the orthogonality properties of infinite sines and cosines.

$$\left. \begin{matrix} C_r^t S_m \\ S_r^t C_m \end{matrix} \right\} = 0 \quad (35)$$

$$\left. \begin{matrix} C_r^t C_m \\ S_r^t S_m \end{matrix} \right\} = 0 \text{ if } r \neq m = +\text{ve constant (k) if } r = m = 0 \quad (36)$$

$$\left. \begin{matrix} C_r^t C_m = 2 \text{ (+ve constant)} \\ S_r^t S_m = 0 \end{matrix} \right\} \text{if } r = m = 0 \quad (37)$$

Using the matrix property of eigenvalues and associated eigenvectors, it can be shown that C_m and S_m are eigenvectors of the autocorrelation matrix T, (in fact they already hold a necessary property-orthogonality).

In general, for a square matrix B, if one writes:

$$Bx = \lambda x \quad (38)$$

λ is an eigenvalue,
x is the associated eigenvector.

Now, since,

Therefore the autocorrelation matrix appears as:

$$A \begin{bmatrix} R(0) & R(1) & \cdots & R(N) \\ R(1) & R(0) & \ddots & \vdots \\ \vdots & \ddots & R(0) & R(1) \\ \cdots & & R(1) & R(0) \end{bmatrix} \quad (19)$$

This matrix is not only square, symmentric and non-negative definite, but is also of toeplitz form [8]. What distinguishes such a matrix is that each diagonal has equal entries, [6].

3. The Autocorrelation Matrix and The Associated Energy Spectrum

The autocorrelation and energy spectrum are a Fourier cosine transform pair.
The autocorrelation maybe expressed as:

$$R(\tau) = \int_0^{\infty} E(\omega) \cos \omega\tau \, d\omega \quad (20)$$

where $E(\omega)$ is the normalised energy spectral density function, which shall be refered to simply, as the energy spectrum, or power spectrum. For ergodic data that $R(\tau) \rightarrow 0$ as $\tau \rightarrow \infty$, $R(\tau)$ from the trapezoidal integration rule, maybe expressed as [2]:

$$R(\tau) = \Delta\omega \left\{ \frac{1}{2} E(0) + \sum_{r=1}^{\infty} E(r\Delta\omega) \cos r\Delta\omega\tau \right\} \quad (21)$$

or equivalently:

$$R(\tau) = \Delta\omega \sum_{r=0}^{\infty} \alpha_r . E(r\Delta\omega) \cos r\Delta\omega\tau \quad (22)$$

where,

$$\alpha_r = 1 \; ; \quad r \geq 1 \quad (23)$$

$$\alpha_r = \frac{1}{2} \; ; \quad r = 0 \quad (24)$$

Then the value of $R(\tau)$, at a particular value of $\tau = p\Delta\tau$, is:

$$R(p\Delta\tau) = \Delta\omega \sum_{r=0}^{\infty} \alpha_r \, E(r\Delta\omega) \cos r_{\Delta\omega} p_{\Delta\tau} \quad (25)$$

ie:

$$R(0) = \Delta\omega \left(\frac{1}{2} E(0) \cos 0 + E(\Delta\omega) \cos 0 + E(2\Delta\omega) \cos 0 + \ldots\right),$$

$$R(\Delta\tau) = \Delta\omega \left(\frac{1}{2} E(0) \cos 0 + E(\Delta\omega) \cos \Delta\omega\Delta\tau + E(2\Delta\omega) \cos 2\Delta\omega\Delta\tau + \ldots\right), \quad (26)$$

$$R(2\Delta\tau) = \Delta\omega \left(\frac{1}{2} E(0) \cos 0 + E(\Delta\omega) \cos 2\Delta\omega\Delta\tau + E(2\Delta\omega) \cos 4\Delta\omega\Delta\tau + \ldots\right),$$

etc., ...
Now, consider a matrix T_r, of Toeplitz form, such that:

$$T_r = \begin{bmatrix} \cos 0 & \cos r\Delta\omega\Delta\tau & \cos 2\Delta\omega\Delta\tau & \cdots \\ \cos r\Delta\omega\Delta\tau & \cos 0 & \cos r\Delta\omega\Delta\tau & \vdots \\ \cos 2r\Delta\omega\Delta\tau & \cos r\Delta\omega\Delta\tau & \ddots & \cos r\Delta\omega\Delta\tau \\ \vdots & \ddots & & \\ & \cdots & \cos r\Delta\omega\Delta\tau & \cos 0 \end{bmatrix} \quad (27)$$

From equations (19), (26) and (27), the autocorrelation matrix A maybe formed:

$$A = \Delta\omega \sum_{r=0}^{\infty} \alpha_r . E(r\Delta\omega) T_r \quad (28)$$

where $E(r\Delta\omega)$ is scalar; that is:

$$A = \Delta\omega \left[\frac{1}{2} E(0) T_0 + E(1) T_1 + E(2) T_2 + \ldots \right] \quad (29)$$

putting $r_{\Delta\omega\Delta\tau} = \phi_r$ in matrix T_r and noting the equality of $\cos(\phi_r)$ and $\cos(-\phi_r)$; consider an element i, j of T_r, (row i, col j):

$$T_r(i,j) = \cos(i-j)\phi_r \quad (30)$$
$$= \cos j\phi_r \cos i\phi_r + \sin j\phi_r \sin i\phi_r$$

godic phenomena.

2. The Autocorrelation Matrix

Given an ergodic phenomenon represented by a time series f (t), whose mean is zero, the autocovariance function is defined as, [2]:

$$\phi(t) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t)\,.\,f(t+\tau)\,dt \qquad (1)$$

and the autocorrelation function as, [2]:

$$R(t) = \frac{\lim_{T\to\infty} \frac{1}{T}\int_0^T f(t)\,.\,f(t+\tau)\,dt}{\lim_{T\to\infty} \frac{1}{T}\int_0^T f(t)\,.\,f(t)\,dt} \qquad (2)$$

This may be written as, [2]:

$$R(\tau) = \frac{\overline{f(t)\,.\,f(t+\tau)}}{\overline{f(t)^2}} \qquad (3)$$

Where the bar represents time averages. In order to approach the autocorrelation matrix, let a function F (t) be given by:

$$\begin{aligned} F(t) &= \alpha_0 f(t) + \alpha_1 f(t+\tau) + \alpha_2 f(t+2\tau) + \dots \\ &= \sum_{n=0}^{\infty} \alpha_n f(t+n\tau) \end{aligned} \qquad (4)$$

Where α_n is an arbitrary coefficient. Also note the fact that [6], [7]:

$$\int_0^T F^2(t)\,dt \geq 0, \text{(for all values of } \alpha_n) \qquad (5)$$

Now, F (t) maybe expressed in matrix form as:

$$F(t) = f_\alpha \qquad (6)$$

Where,

$$F = [f(\tau), f(t+\tau), f(t+2\tau), \dots] \qquad (7)$$

and,

$$\alpha^t = [\alpha_0 \quad \alpha_1 \quad \alpha_2 \quad \dots] \qquad (8)$$

In which case, the integral in (5), can be manipulated [7] as:

$$\int_0^T F^2(t)\,dt = \int_0^T \alpha^t F^t F \alpha\,dt \qquad (9)$$

Where α^t and F^t refer to α transpose and F transpose. The inequality (5) may, therefore, be re-written [7] as:

$$\int_0^T F^2(t)\,dt = \alpha^t \left[\int_0^T F^t \; F \; dt\right] \alpha \geq 0, \text{(for all } \alpha) \qquad (10)$$

This is known as a quadratic form, with real symmetric, non-negative definite, matrix [7], A,

$$A = \int_0^T F^t \; F \; dt \qquad (11)$$

Considering the time averaged elements of matrix A; element i, j (row i, col j):

$$A_{ij} = \overline{f(t+k\tau)\,.\,f(t+l\tau)} \qquad (12)$$

where,

$$i, j = 1, 2, 3 \dots \qquad (13)$$

and,

$$k = i - 1, \quad l = j - 1 \qquad (14)$$

Equation (12) may be equivalently written as:

$$A_{ij} = \overline{f(t)\,.\,f(t+n\tau)} \qquad (15)$$

where,

$$n = l - k = j - i \qquad (16)$$

Normalising the elements of this matrix, by dividing through by element A_{ii} yields:

$$A_{ij} = \frac{\overline{f(t)\,.\,f(t+n\tau)}}{\overline{f(t)^2}} \qquad (17)$$

which is an autocorrelation coefficient; for time interval $\Delta\tau$:

$$A_{ij} = R(n \;\; \Delta\tau) \qquad (18)$$

On the Determinant And Eigenvalues of the Autocorrelation Matrices

Nezameddin Faghih

Shiraz University, Iran

Abstract:

The autocorrelation function theorem for ergodic phenomena, is used to derive the autocorrelation matrix and show that the autocorrelation coefficients maybe represented as such a non- negative definite matrix of Toeplitz form. The properties of this matrix and its sub-matrices are discussed. It is proved that the eigenvalues of the infinite autocorrelation matrix are proportional to the energy spectral densities at their corresponding frequencies and vice versa; these eigenvalues are real and non-negative. It is also shown that the determinant of the autocorrelation matrix is always bounded between zero and unity; which reveals further characteristics of the autocorrelation matrix of ergodic phenomena.

1. Introduction

The autocorrelation function is regarded as an invaluable intermediate stage in spectral analysis of ergodic phenomena, [1]. The autocorrelation function, for ergodic data, is a measure of time- related (or space-related) properties in the data that are separated by fixed time (or space) intervals, [2]. It can be estimated by shifting the ergodic data record relative to itself by some fixed time (or space) intervals, then multiplying the original record, with the shifted record. and averaging the resulting product values over the available record length. This procedure can be repeated for various intervals, [2].

The energy spectrum or power spectral (also called autospectral) density function describes the variation of mean square value with frequency. It can be estimated by computing the mean square value of ergodic data, in a narrow frequency band, at various center frequencies, and then deviding by the frequency band, [2]. In practice, however, the spectral estimation is approached by estimating the autocorrelation function, [2], [3]. In fact, the autocorrelation function and the spectral density function are related by the fourier cosine transform, known as the Wiener-Khinchine relations, [4]. For digital computer spectral estimation, a discrete truncated (due to limitations on availability or acquisition of data, estimation errors, etc.) autocorrelation function and some numerical integration method. most usually trapezoidal rule, are used, [2], [5]. This can cause various problems and may even result in misleading or even theoretically unacceptable spectral estimations, for example attenuated or negative spectra, [2].

Therefore, investigation of the autocorrelation characteristics, and its relation to spectral estimations,is a non-trivial problem requiring enthusiastic research. This paper considers the properties of the autocorrelation matrix and its characteristic values, ie the eigenvalues and eigenvectors, and the relation to the associated power spectrum. The determinant of the autocorrelation matrix is also considered and its bounds are investigated. It is hoped that the work reported in this paper will pave for further research and development of spectral estimation methods, and that it renders further illustrations on the behaviour of autocorrelations and, hence, er