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Proof: It is omitted

III. TIME VARYING SYSTEM:

Case1: $R \neq 0$

Assume $E(t)x^{o}(t)=A(t)x(t)+B(t)u(t)$ is given. By some modification we will get a singular equation similar to (8).

$$\begin{bmatrix} E(t) & 0 \\ 0 & E(t)^T \end{bmatrix}\begin{bmatrix} X^o \\ Z^o \end{bmatrix}=\begin{bmatrix} A(t)+E^o(t) & -B(t)R^{-1}B^T(t) \\ -Q & -(A(t)+E^o(t))^T \end{bmatrix}\begin{bmatrix} X \\ Z \end{bmatrix} \quad (9)$$

Case2: R= 0

In this case we will have:

$$\begin{bmatrix} E(t) & 0 \\ 0 & E^T(t) \end{bmatrix}\begin{bmatrix} X^o \\ Z^o \end{bmatrix}=\begin{bmatrix} A(t)+E^o(t)+B(t)P_2 & B(t)P_1 \\ -Q & (A(t)+E^o(t))^T \end{bmatrix}\begin{bmatrix} X \\ Z \end{bmatrix} \quad (10)$$

Remark: If E=I, and R=0 then $P_1=-R^{-1}B^T$, $P_2=0$ and we will get the same Riccati equation as we had for non singular system.

$$x^o(t)=Ax(t)+Bu(t)$$

Example:

Given

$$J(x,u)=\frac{1}{2}\int_0^\infty (x^T(t)Qx(t)+u^T(t)Ru(t))\,dt$$

Subject to

$$\begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix}\begin{bmatrix} x_1^o \\ x_2^o \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}+\begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t)$$

Where

$$Q=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad R=1 \quad , \quad x(0)=x_o$$

First we solve it as "Non singular system" and after finding the solution we let $\varepsilon \to 0$. Second we will use above procedure in order to get the solution and finally we will compare the answers.

Case1: Multiplying by $\begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix}^{-1}$ will give us

$$\begin{bmatrix} x_1^o \\ x_2^o \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 1/\varepsilon & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}+\begin{bmatrix} 0 \\ 1/\varepsilon \end{bmatrix}u(t)$$

By solving Riccati equiation we obtion,

$$P_{12}=\frac{1 \pm \sqrt{2}}{\varepsilon} \qquad P_{22}=\pm\frac{\sqrt{2\varepsilon(1 \pm \sqrt{2})}}{\varepsilon}$$

and

$$u(t)=-R^{-1}B^TPX(t)=-\varepsilon P_{12}X_1(t)-\varepsilon P_{22}X_2(t)$$

$$=(-1 \pm \sqrt{2})\,X_1(t) \pm \sqrt{2\varepsilon(1 \pm \sqrt{2})}\;X_2(t)$$

as $\quad \varepsilon \to 0$

$$u(t)=(-1 \pm \sqrt{2})\,X_1(t)$$

Case 2: that is

$$E=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad u(t)=-R^{-1}B^TPEx(t)=-P_{12}X_1(t)$$

thus, $u(t)=-(1 \pm \sqrt{2})\,X_1(t)$ that is we got the same answers.

Conclusion:

Linear singular optimal regulator problem was discussed. We dierived the generalized Riccati equation for both time-invarient and time varying cases. We showed $P_{12}=1 \pm \sqrt{2}$ that for case $R \neq 0$, the Riccati equation is symetric.

However, when R=0 , we will not have symetric property for matrix P.

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$$\left[E^TP^o(t)\,E + E^TP(t)\,A - E^TP(t)\,BR^{-1}B^TP(t)\,E + A^TP\,(t)\,E + Q\right] x(t) = 0$$

Since this holds for all admissible x(to) and therefore all admissible x(t), Thus

$$E^TP^o(t)\,E + E^TP(t)\,A - E(t)P(t)\,BR^{-1}B^TP(t)\,E + A^TPE + Q = 0$$

where,

$$E^TP(t_f)\,E = E^TSE$$

and this is generalized Riccati equation.

The deseriptor feed back control is

$$u(t) = -Kx(t) \qquad (6)$$

On the other hand we have

$$Ru(t) + B^T\lambda(t) = 0$$

Therefore

$$u = -R^{-1}B^T\lambda(t) = -R^{-1}B(t)P(t)Ex$$

that is

$$K = R^{-1}B^TP(t)E$$

substituting (6a) in (2a) we will get

$$\begin{bmatrix} E & 0 & 0 \\ 0 & E^T & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x^o \\ \lambda^o \\ u^o \end{bmatrix} = \begin{bmatrix} (A-BK) & 0 & 0 \\ -Q & -A^T & 0 \\ 0 & B^T & R \end{bmatrix}\begin{bmatrix} x \\ \lambda \\ u \end{bmatrix}$$

Thuse we can state the following theorem.

THEOREM (3):

System

$$Ex^o(t) = (A-BK)x(t) + BV(t)$$

is Regular if Hamiltoniam system is Regular.

proof:

$$|Es - A| = |SE - (A-BK)|\,|SE^T + A^T|\,|R|$$

by looking at above expression if $|Es - A| \neq 0$ then, $|SE - A + BK| \neq 0$ and we are done.

Remark

If $|R|=0$, we can not find u interms of λ directly. However in [8] it is shown that $u=P_1\lambda+P_2x$ for some P1 and P2. By substituting this expersion for u in proceses of deriving Riccati equation we will obtian.

$$E^TP^o(t)E+E^TP(t)A+E^TP(t)BP_1p(t)E+E^TP(t)BP_2+A^TP(t)E+Q=0$$

This is not symetric

II. SOLUTION OF RICCATI EQUATION:

Let us assume, $PEX=Z$ and $E^TPEX=E^TZ$. where X, Z are Matrices.

By taking derivitive, $E^TP^oEX + E^TPEX^o=E^TZ^o$ substitute.for E^TP^oE from Riccati equation then.

$$-E^TP(AX-BR^{-1}B^TZ-EX^o)-(E^TZ^o+QX+A^TZ)=0$$

If both parenthesis are zero, the equation is satisfied. therefore we have the pair of linear differential equations:

$$\begin{aligned} EX^o &= AX-BR^{-1}B^TZ \\ E^TZ^o &= -QX-A^TZ \end{aligned} \qquad (7)$$

Or

$$\begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix}\begin{bmatrix} X^o \\ Z^o \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}\begin{bmatrix} X \\ Z \end{bmatrix} \qquad (8)$$

with $X(t_f)=I$, $E^TZ(t_f)=E^TSE$

Equation (8) has unique solution if and only if

$$\Delta(\lambda) = \begin{bmatrix} \lambda E-A & BR^{-1}B^T \\ Q & \lambda E^T+A^T \end{bmatrix} \neq 0$$

for some λ for soultion see[1].

For existence and uniqueness of equation (8), we will state the following theorem:

THEOREM(4):

Equation (8) is regular if and only if equation (1) is regular.

$$\frac{\partial H}{\partial u} = 0 = Ru + B^T\lambda \quad (2a)$$

$$-\frac{\partial H}{\partial x} = E^T\lambda^o = -Qx - A^T\lambda \quad (2b)$$

$$Ex^o = Ax + Bu \quad (2c)$$

or

$$\begin{bmatrix} E & 0 & 0 \\ 0 & E^T & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x^o \\ \lambda^o \\ u^o \end{bmatrix} = \begin{bmatrix} A & 0 & B \\ -Q & -A^T & 0 \\ 0 & B^T & R \end{bmatrix} \begin{bmatrix} x \\ \lambda \\ u \end{bmatrix} \quad (2)$$

or

$$Ex^o = \Lambda x$$

with

$$x(t_o) = x_o \quad , \quad E^T\lambda(t_f) = E^TSEx(t_f)$$

We assume equation (1) is regular that is there exist a unique solution x(t) for all $x(t_o)$ in the subspace of admissible initial condition H [6]. We can solve these trajectories by two methods.

METHOD(1):

Case(1) $|R| \neq 0$

Find $U = R^{-1}B^T\lambda$ and substitute for it in other equation.
Then we have,

$$\begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} \begin{bmatrix} x^o \\ \lambda^o \end{bmatrix} = \begin{bmatrix} A & BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \quad (3)$$

That is we need to solve above singular equation. The existance and uniqueness for solution are given in next theorm.

THEOREM(1)

Equation (3) is tractable [1] if and only if equation (1) is tractable (Regular).

PROOF:

By some manipulation we obtain.

$$\Delta = \begin{vmatrix} SE-A & -BR^{-1}B^T \\ Q & SE^T+A^T \end{vmatrix} = |SE-A| \left|(SE^T+A^T) + BR^{-1}B^T(SE-A)^{-1}Q\right|$$

$$= |SE-A| \, |(SE+A)^T| \, |I + BR^{-1}B^T(SE-A)^{-1}Q(SE+A)^{-1}|$$

$$= \Delta_1 \Delta_2 \Delta_3$$

Only if part

It is easily seen that $\Delta \neq 0$ implies $\Delta_1 \neq 0$

If part

$\Delta_1 \neq 0$ implies $\Delta_2 \neq 0$ and since $\Delta_3 \neq 0$ therefore $\Delta \neq 0$. Thus by [1], we are done.

Case(2) $|R|=0$

Then we can not find u directly from equation (2a). Thus we solve equation (2) directly. Easily it can be shown that [7].

$$|Es - A| = (-1)^n |SE - A| \, |(SE + A)^T| \, |R + H^T(-s) H(s)|$$

Where

$$H(s) = \sqrt{Q}\,(SE - A)^{-1}B$$

This leads us to the following theorem.

THEOREM(2):

Equation (2) is tractable if and only if equation (1) is tractable.

PROOF:

Similar to theorem (1) therefore it is omitted.

METHOD(2) (RICCATI Equation)

Assume $\lambda(t) = p(t)\,Ex(t)$

then

$$\lambda^o(t) = P^o(t)\,Ex(t) + P(t)\,Ex^o(t)$$

and

$$E^T\lambda^o(t) = E^TP^o(t)\,Ex(t) + E^TP(t)\,Ex^o(t)$$
$$= E^TP^o(t)\,Ex(t) + E^TP(t)\left[Ax(t) + Bu(t)\right] \quad (4)$$

We also have

$$E^T\lambda^o(t) = -Qx(t) - A^T\lambda(t) = -Qx(t) - A^TP(t)\,Ex(t) \quad (5)$$

If we set equation (4) and (5) equal to each other, we get

$$-Qx(t) - A^TP(t)\,Ex(t) = E^TP^o(t)\,E\,x(t) + E^TP(t)\,Ax(t) + E^TP(t)\,B\left[-R^{-1}B^TP(t)\,Ex(t)\right]$$

or

The Linear-Quadratic Optimal Regulator Problem For Continuous-Time Descriptor Systems

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Abstract

Descriptor systems have been the subject of recent interest due to their many practical applications. This paper considers the linear quadratic optimal regulator problem for the descriptor system. Hamilton-Jacobi theory is applied in order to compute the optimal control and associated trajectory. Two methods are presented for solving these trajectories. The first method uses the concept of the Drazin inverse. The second method involves the derivation and solution of a Riccati equation. Necessary and sufficient conditions for the existence and uniqueness of a solution are presented. Examples are provided to illustrate the technique.

INTRODUCTION:

This paper will discuss the linear-Quadratic optimal Regulator problem for the continuous time-invarient descriptor system,

$$Ex(t) = Ax(t) + Bu(t) \qquad (1)$$

Where the matrix E is assumed to be singular and

$$x \in R^n \ , \ u \in R^m \ , \ y \in R^p$$

Systems described by equation (1) are known as singular systems [1], semi-state system [2], Generalized state space systems [3], or descriptor systems [4]. They consist of both static and dynamic equations and need not to be causal. We assume that (SE-A) is regular that is $|SE-A| \neq 0$ where $|.|$ denotes the determinant. This condition casuses that for appropriate initial condition the equation (1) has a solution [5].

I. THE LINEAR REGULATOR FOR SINGULAR SYSTEMS:

Given that

$$Ex^{\circ}(t) = Ax(t) + Bu(t) \quad , \ x \in R^n \ , \ u \in R^m$$

We wish to find the control which minimizes the cost founction.

$$J = \frac{1}{2} x^T(t_f) E^T S E x(t_f) + \frac{1}{2} \int_{t_o}^{t_f} [x^T(t) Q x(t) + u^T R u(t)] \, dt$$

Where t_f is fixed, $x(t_f)$= free, $x(t_o)=x_o$, Q, R, S are symmetric, and at least P.S.d., $|R| \neq 0$, and $|E|=0$.

By defining Hamiltonian equation and using the calculus of variations, we will obtain the following:

$$H[x(t), u(t), \lambda(t), t] = \frac{1}{2} x^T(t) Q x(t) + \frac{1}{2} u^T(t) R u(t) + \lambda^T [Ax(t) + Bu(t)]$$

and

$$J = \frac{1}{2} x^T(t_f) E^T S E x(t_f) + \frac{1}{2} \int_{t_o}^{t_f} [x^T(t) Q x(t) + u^T(t) R u(t) + \lambda^T (Ax(t) + Bu(t) - Ex^{\circ}(t))] dt$$

$$= \frac{1}{2} x^T(t_f) E^T S E x(t_f) + \frac{1}{2} \int_{t_o}^{t_f} [H - \lambda^T E x^{\circ}(t)] dt$$

$$= [\frac{1}{2} x^T(t_f) E^T S E x(t_f) - \lambda^T E x(t_f)] + \int_{t_o}^{t_f} [H - \lambda^{\circ T} E x(t)] dt$$

then

$$\partial J = \partial x(t_f)(E^T S E - E^T \lambda) + \int_{t_o}^{t_f} [\partial x^T (\frac{\partial H}{\partial x} + E^T \lambda^{\circ}) + \partial u^T (\frac{\partial H}{\partial u})]$$

For minimum, it is necessary that the first variation in J vanish for arbitrary dx, and du. Thus we get .