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may result in an unstable model (if the coefficient of s^2 in eq. 24 becomes negative). The influence of the cancelled pole-zero pair is rather complex and is given by eqs. 17 and 22.

The influence of the system parameters on the damping ratio of the reduced order model is also important in the study of the transient behavior and frequency response of the model. The most important specifications, in both time and frequency domains, for the second order system is given by [5].

(present overshoot) $P\,O = 100 \exp\left\{ -\pi\zeta_r / \sqrt{1-\zeta_r^2} \right\}$ (25)

(peak time) $t_p = \dfrac{\pi}{\omega_{nr}\sqrt{1-\zeta_r^2}}$ (26)

(settling time) $t_s = \dfrac{k}{\omega_{nr}\xi_r}$ (27)

(rise time) $t_r = \dfrac{7.04\,\xi_r^2 + 0.2}{\omega_{nr}\,\xi_r}$ (28)

(delay time) $t_d = (1 + 0.7\,\xi_r) / \omega_{nr}$ (29)

(M peak) $M_p = 1 / (2\xi_r\sqrt{1-\xi_r^2})$ (30)

(N peak) $W_f = \omega_{nr}\sqrt{1-2\xi_r^2}$ (31)

(bandwidth)

$$BW = \omega_{nr}\sqrt{1-2\xi_r^2+\sqrt{1+(1-2\xi_r^2)^2}} \quad (32)$$

After normalizing, these specifications ultimately depend on ξ_r. Let us consider the percent overshoot as a measure of relative stability and bandwidth as a measure of response speed. It is known that for the third order system characterized by eq. 13, if we fix K_v and ξ, both measures increase with increasing λ, reach a maximum, and then decrease as the complex poles become dominant and reach an asyptotic final value [5]. These results are confirmed by eq 22 which shows that ξ_r reaches a minimum (when the system is not too underdamped) at $\lambda = K_v$. Also, it can be seen that when K_v or ξ is changed, the specifications of the system and its model change in the same direction. This match between the system and model specifications becomes better as the values of λ, δ, and ξ are increased.

Conclusion

The investigation carried out in this paper confirms the theoretical arguments concerning the effectiveness of the Pade method (2). It has been shown that while the Pade method should ideally be used when simulating the steady-state behavior of the system or carrying out accuracy computations, in most cases the Pade approximants are also capable of simulating the transient behavior and the frequency response fairly well; especially when the input process does not have too abrupt changes or the input frequency is not large. However, it is very important to know when the Pade approximants fail to simulate the behavior of the given system. It has been demonstrated in this paper that this may be the case when there are small zeros, or the system is too underdamped, or the dominant poles are not located close together with enough distance from the non-dominant poles that are to be eliminated.

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$$BW = \sqrt{1 + \frac{\omega_n^2}{\delta^2} - 2\zeta^2 + \sqrt{1 + (1 + \frac{\omega_n^2}{\delta^2} - 2\zeta^2)^2}} \quad (11)$$

$$BW' = \frac{1}{2\zeta - \omega_n / \delta} \quad (12)$$

It is seen that both bandwidths tend to decrease with decreasing damping ratio ξ and/or increasing δ.

The above analysis, while revealing the main features of the Pade method has an important drawback. The second order transfer function given by eq. 1 is not very typical. Most practical systems involve a power element with inertia, which yeilds a pole-zero excess of 2. Therefore, we go on to consider the tranfer function

$$H(s) = \frac{K(s+\delta)}{(s+\lambda)(s^2 + 2\zeta\omega_n s + \omega_n^2)} \quad (13)$$

Not only is this more representative of practical systems due to the increased complexity and pole-zero excess of 2, but its consideration also allows us to study the influence of the extra pole especially when λ and ω_n become comparable. Unfortunately, we do not have close-form formulas for a third order system's specifications. However, the characteristics of these systems are well known [5]. Again, we chose the gain

$$K = \frac{\lambda \omega_n^2}{\delta} \quad (14)$$

So that we have $K_p = \infty$

The Pade approximant is given by

$$H_r(s) = \frac{\omega_{nr}^2}{s^2 + 2\zeta_r \omega_{nr} s + \omega_{nr}^2} \quad (15)$$

Where

$$\omega_{nr} = \frac{\omega_n}{[1 + (2\zeta - \omega_n/\delta)(\omega_n/\lambda - \omega_n/\delta)]^{1/2}} \quad (16)$$

$$\zeta_r = \frac{2\zeta + \omega_n/\lambda - \omega_n/\delta}{2[1 + (2\zeta - \omega_n/s - \omega_n/\lambda - \omega_n/\delta)]^{1/2}} \quad (17)$$

The first three static error coefficients are equal for both the system and its reduced order model

$$K_p = \infty \quad (18)$$

$$K_r = \frac{1}{2\zeta/\omega_n + 1/\lambda - 1/\delta} \quad (19)$$

$$K_a = \frac{1}{\frac{1}{\omega_n^2} - \frac{1}{\lambda}(\frac{1}{\lambda} - \frac{1}{\delta}) - \frac{2\zeta}{\omega_n}(2\zeta/\omega_n + 1/\lambda - 1/\delta)} \quad (20)$$

or equivalently, expressing the above equations in terms of K_v we obtain

$$\omega_{nr}^2 = \frac{1}{\frac{1}{\omega_n^2} + (1/K_r - !/\lambda)(1/K_r - 2\zeta/\omega)} \quad (21)$$

$$\zeta_r = \frac{1/K_r}{2[1/\omega_n^2 + (1/K_v - 1/\lambda)(1/K_v - 2\zeta/\omega_n)]^{1/2}} \quad (22)$$

$$K_a = \frac{1}{1/\omega_n^2 - 1/\lambda(1/K_v - 2\zeta/\omega_n) - 2\zeta/\omega_n \; 1/K_v} \quad (23)$$

$$H_r(s) = \frac{1}{[1/\omega_n^2 + (1/K_v - 1/\lambda)(1/K_v - 2\zeta/\omega_n)]s^2 + (1/K_v)s + 1} \quad (24)$$

The stability is rpeserved if K_v and K_a are both positive. For K_v to be positive the system should not be too underdamped. But here we also have another thing to worry about. If λ is too small, especially when we have large damping ratio and a distant zero, then K_a may become negative (see eqs. 20, 23). On the other hand, it can be seen that reduced order model is now capable of simulating the undrdamped behavior of the system if there are dominant complex poles and the damping ratio ξ is small enough. But too small damping ratios assume so that we have $K_p = \infty$.

The Pade approximant is given by

$$H_r(s) = \frac{1}{1 + (2\zeta / \omega_n - 1/\delta)\, s} \qquad (3)$$

The first two static error coeficients are equal for both systems

$$K_p = \infty \qquad (4)$$

$$K_v = \frac{1}{2\zeta/\omega - 1/\delta} \qquad (5)$$

The existence of the zero gives us the necessary flexibility for controlling the velocity error of the systems, it is to be noted that the ability of automatic devices to perform as required is often dependent to a large extent on their ability to follow constant-velocity inputs [5]. Zero velocity error can be obtained by setting

$$\omega_n = 2\zeta\delta \qquad (6)$$

If the value of the damping ratio ξ is chosen less than that indicated by eq 6($\omega_n/2\delta$) ie if the system is too underdamped or the zero is too close to the origin, then the time constant $\tau = 1/K_v$ becomes negative and the system responds to ramp input with a lead rather than lag. It is in this case (and only in this case) that the Pade method fails to retain the stability of the given system because there is no stable first order system that can simulate this property of the second order systems. The Pade method has also been criticized for failing to cancel almost identical pole-zero pairs [6]. The simple system considered here allows us to see how this can happen. Suppose the system is overdamped, so that it has two distinct real poles, and has a zero very close to the dominant pole. If the poles are designated as λ_1 and λ_2 then from eq 5 we have

$$1/K_v = 1/\lambda_2 + (\delta - \lambda_1)/\delta\lambda_1 \qquad (7)$$

Cancelling out δ and λ_1 leaves out the second term in eq 7, which, however, may be significant if the pole-zero pair is too close to the origin. In other words, the Pade method does not cancel the pole-zero pair whenever this would cause a significant change in the steady-state behavior of the system

The transient responses of the first and second order systems given by eqs. 1 and 2 are naturally different. The zero tends to shift the approximant's pole to the right and the closer it is to the origin the more it displaces the pole. Also, the first order approximant cannot simulate the oscillatory transient behavior of the system when it is underdamped, and small damping ratios tend to shift the approximant's pole too much to the left making it respond to jump inputs much quicker than the given system. On the positive side, these displacements tend to cancel each other. The transient response of the system is fairly well simulated when the system is overdamped and the zero is not too small. As for the frequency responses, they are given by

$$H(j\omega) = \frac{1 + j\omega/\delta}{1 - (\omega/\omega)^2 + j\,2\zeta\,\omega/\omega_n} \qquad (8)$$

$$H_r(j\omega) = \frac{1}{1 + j\,(2\zeta\,\omega/\omega_n - \omega/\delta)} \qquad (9)$$

These equations show how the shifts in the approximant's pole occur in such a way to compensate for the errors caused by the cancellation of the extra pole and zero subtracting eq. 9 from eq. 8 we get

$$H(j\omega) - H_r(\omega) = \frac{\left(\frac{\omega}{\omega_n}\right)^2 \left(2\zeta\frac{\omega_n}{\delta} - \frac{\omega_n^2}{\delta^2} - 1\right)}{1 - \left(\frac{\omega}{\omega_n}\right)^2 \left(1 + \frac{2\zeta\omega_n}{\delta} - 4\zeta^2\right) + j\frac{\omega}{\omega_n}\left[\left(1 - \frac{\omega^2}{\omega_n^2}\right)\left(2\zeta - \frac{\omega_n}{\delta}\right) + 2\zeta\right]} \qquad (10)$$

The difference increases when the damping ratio becomes too small or when the zero is too close to the origin, and vanishes when the zero coincides with one of the poles. The bandwidths of the system and its reduced order model are given by upon the class of inputs likely to be applied as well as whether the steady-state or the transient behavior of the system is to be simulated. When the purpose of reduced order modelling is to carry out analysis or design computations, which cannot easily be carried out for less approximate but more complex models of the system, it is very important to be able to predict the effectiveness of the reduction method for general classes of problems rather than particular instances. The Pade method is currently the most popular technique for order reduction, and naturally, it has been compared with the other techniques in several different papers. Very briefly, the reduced order model is chosen so that as many terms of the power series expansion (Mac Laurin series, Taylor series, or several series expansions around different points) of its transfer function coincide that of the given system as the degrees of freedom determined by the choice of the reduced order and the number of zeros will allow. The more common case of equating the Mac Laurin series expansion coefficients is especially important, and will be the subject of our investigation in this paper, because it guarantees the retention of the first time moments of the system [3]. The power series expansion around $S = \infty$ is also important because it leads to an approximant that retains the Markov parameters of the system. It is, therefore, usually used when the simulation of the transient behavior is intended. (A good match around $S = \infty$ in the Laplace domain guarantees a good match around $t = 0$ in the time domain due to the initial value theorem). However, it has been argued that a good simulation of the initial response is not enough for a good or satisfactory simulaiton of the transient response and matching the Markov parameters, rather than retaining the dominant poles, tends to retain the poles most distant from the origin [3].

The usual Pade method based on matching Mac Laurin series expansion coefficients has maximum effectiveness, when the simulation of the steady - state response of the system to inputs having only significant low - order power series expansion terms is intended [2].

The errors introduced by order reduction is usually no more (and no less) than those introduced by approximating the input signal through step, ramp, parabola, and higher-term functions. Thus, for example, it can be ideally utilized in carrying out accuracy compulations. As for simulating the transient behavior and the frequency response of the system, it has been pointed out that although in most cases the Pade method has a quite satisfactory performance [3], [4], there are certain cases where the Pade approximants fail to simulate the behavior of the system [1], [2]. For example, a well known and annoying aspect of the Pade approximation method is that it does not retain the stability of the system. The power of this paper is to carry out an investigation of these cases and compare empirical results with theoretical predictions.

Pade Approximants of Simple Systems

It has been argued that the Pade approximant performs poorly when the chosen reduced order is too small to be able to simulate all the different aspects of the complex system's behavior [3]. This can be demonstrated by considering the Pade approximant of a second order system where transfer function is

$$H(s) = \frac{K\,(s+\delta)}{s^2 + 2\zeta\,\omega_n s + \omega_n^2} \qquad (1)$$

This system has enough degrees of freedom (adjustable) parameters to allow the specification of performance measures describing relative stability, accuracy, and speed of response [5]. Furthermore, it also allows us to study how complex poles and/or Zeros influence the performance of the Pade approximants. Thus the theoretical arguments about their potentially adverse effects on the performance of Pade approximants can be investigated emperically. Since the approximant's performance is not affected by the gain factor K, we

$$K = \frac{\omega_n^2}{\delta} \qquad (2)$$

The Performance of the Pade Approximants: A Theoretical Investigation

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ABSTRACT

The problem of modelling complex systemsvia reduced order approximants has found many applications in the analysis and design of control systems .Several different methods ahve been suggested for reducing the order of a given system .The choice of the best method, however, depends on the particular aspects of the given system's behavior which are to be simulated. For any specific method there can be found cases where the reduced order models 3 behaviors fail to match those of the given systems. It is very important, therefore, to be able to predict the performance of the reduced order aproximant before actually carrying out the calculations necessary for deriving the model. The purpose of this paper is to investigate the performance of the Pade method, which is one of the most commonly used routines for order reduction. After a systematic discussion of recent theoretical findings about the behavior of Pade approximations, these findings are examined empirically. The conditions under which there can be mismatch between the system and model behaviors are discussed.

Introduction

Increased attention has been devoted in recent years to the problem of modelling large-scale or complex systems via suitable reduced order approximants. So many different methods have been suggested for obtaining a reduced order model for any given system that their classification and comparative analysis has become a separate area of investigations [1], [2]. Each month, there appear several papers providing examples that show their methods yeild better approximants than the competing methods. However, all the different reduced order models of a given system obtained by the application of various reduction methods are essentially approximations of that system and can simulate certain aspects of the behavior of the given system better than the other aspects of that system's behavior. Thus the effectiveness of the order reduction method depends