[2] J. R. Cannon, Determination of the unknown coefficient K(u) in the equation $\nabla . K(u) \nabla u = 0$ from overspecified boundary data, J. Math. Anal. Apple. 18, 112-114, 1967.
[3] J. R. Cannon and P. DuChateau, An inverse problem for a nonlinear diffusion equation, SIAM J. Appl. Math. 39(2), 272-289, 1980.
[4] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations, Springer-Verlag, Berlin, 1977.
[5] A. Shidfar and H. Azary, An inverse problem for a nonlinear diffusion equation, J. Nonlinear Analysis, Theory, Methods & Application 28 (4), 589-593, 1997.
[6] P. DuChateau and W. Rundell, Unicity in an inverse problem for an unknown reaction term in a reaction diffusion equation, J. Differential Equations 59, 155-164, 1985.
[7] M. Pilant and W. Rundell, An inverse problem for a nonlinear elliptic differential equation, SIAM J. Math. Anal. 18(6), 1801-1809, 1987.
[8] J. Sylvester and G. Uhlmann, A global uniqueness theorem for an inverse boundary value problem, Ann. of Math. 125, 153-169, 1987.
[9] S. Timoshenko and S. Woinowsky-Krieger, Theory of Plates and Shells, McGrow-Hill, New York, 1959.
[10] A. Shidfar and A. M. Shahrezaee, An inverse nonlinear biharmonic problem, proceedings of 31st Iranian Maths. Conf., 116-121,2000.
[11] C. T. Wang, Applied Elasticity, McGrow-Hill, 1953.
[12] R. Haberman, Elementary Applied Partial Differential Equations, Prentice-Hall Int., 1987.
[13] M. Katsurada and H. Okamoto, The collocation points of the fundamental solution method for the potential problem, Computers Math. Applic. 31(1), 123-137, 1997.
[14] A. Bogomolny, Fundamental solutions method for elliptic boundary value problems, SIAM J. Numer. Anal. 22(4), 644-669, 1986.
[15] Y. Jeon, An indirect boundary integral equation method for the biharmonic equation, SIAM J. Numer. Anal. 31(2), 461-476, 1994.
[16] J. Sylvester and G. Uhlmann, A uniqueness theorem for an inverse boundary value problem in electrical prospection, Comm. Pure Appl. Math. 39, 91-112, 1986.
[17] J. R. Cannon, The One-Dimensional Heat Equation, Addison-Wesley, Menlo Park, California, 1984.
[18] P. DuChateau, Monotonicity and uniqueness results in identifying an unknown coefficient in a nonlinear diffusion equatin, SIAM J. Appl. Math. 41(2), 310-323, 1981.
[19] P. Linz, Theoretical Numerical Analysis, John Wiley & Sons, 1979.
[20] H. Adibi and R. Masoumi, A Fresh Look at 2-D Point Forces, Int. J. of Eng. Sci., 9(1), 73-82, 1998.
[21] H. Adibi, Some 2-D biharmonic formulations, Ph. D. thesis, City University, London, U. K. 1989.

$$\nabla^2 V(x,y) = 0 \quad in\ \Omega, \tag{25}$$

$$\frac{\partial V}{\partial n}(x,y) = 0 \quad on\ \partial\Omega_2, \tag{26}$$

$$V(x,y) = \begin{cases} \int_0^{f_3(s_1)} D(\eta)d\eta & on\,\partial\Omega_1, \quad (27) \\ \int_0^{f_1(s_2)} D(\eta)d\eta & on\,\partial\Omega_2. \quad (28) \end{cases}$$

Using the strong maximum principle, $V(x,y)$ may not attain its maximum in the interior of Ω or on the arc $\partial\Omega_2$ where $\frac{\partial V}{\partial n} = 0$. Therefore the maximum values of $V(x,y)$ on $\overline{\Omega}$ must lie in the range of the conditin (27) for $s_1 \in \partial\Omega_1$. This assumption implies that the range of $V(x,y)$ must lie in the range of values $V(x,y)$ defined by (28) for $s_2 \in \partial\Omega_2$. The continuity of $f_1(s_2)$ then demands that $V(x,y)$ must attain its maximum on $\partial\Omega_2$, which may only happen if $V(x,y)$ is constant. Since both of $f_1(s_2)$ and $f_3(s_1)$ may not be constant functions. Thus, we conclude that $V(x,y)$=0, and from (28) the function $D(w)$ must be zero for any w in the range of f_1. This completes the proof of theorem.

3-Conclusions

If f_1 and f_3 are both strictly monotonic functions on their domains and continuous at the endpoints (x_0, y_0) and (x_1, y_1) that implies that $\text{range}_{\partial\Omega_2} f_1 = \text{range}_{\partial\Omega_1} f_3$, we find that there is at most one solution for the inverse problem (1)-(5). The mapping K is a bounded positive operator from the space of $C^1(\partial\Omega_1)$ to $C^1(\partial\Omega_2)$, in fact $\|K\|_\infty = 1$, where $\|...\|_\infty$ denotes the supremum operator norm.

To see this, note that for any g(s) continuous on $\partial\Omega_1, K\{g\}$ represent the value of the solution of Laplace equation on the segment of the boundary $\partial\Omega_2$, where $\frac{\partial V}{\partial n} = 0$. As in the proof of theorem, the maximum principle shows that [19]

$$\|K\|_\infty = \frac{\sup_{\partial\Omega_2} |K[g(s)]|}{\sup_{\partial\Omega_1} |g|} \le 1. \tag{29}$$

Equality follows from the fact that if $g = g^{(0)}$ for some constant $g^{(0)}$ then $K[g^{(0)}] = g^{(0)}$. This shows that if constant functions are admissible then 1 is in the spectrum of K, that is, $\frac{\partial G^*}{\partial n}$, has a singularity of the order or $[(x-\xi)^2 + (y-\eta)^2]^{-1}$. Due to the difference in the arguments of the kernel of the linear transformation (22), T will not in general be a symmetric operatior.

References

[1] Y. Matsuzawa, Finite element approximation for some quasilinear elliptic problems, J. of Comp. and Appl. Maths. 96, 13-25, 1998.

Thus, from (18) and the overspecified condition (17), we find

$$\int_0^{f_1(s_2)} d(\eta)\,d\eta = \iint_\Omega G^* M \,d\xi\, d\eta - \int_{\partial\Omega_2} G^* f_2\, ds_2 + \int_{\partial\Omega_1} \frac{\partial G^*}{\partial n} . \left(\int_0^{f_3(s_1)} D(\eta)d\eta\right) ds_1 . \tag{19}$$

Putting

$$\Psi = \iint_\Omega G^* M\, d\xi\, d\eta - \int_{\partial\Omega_2} G^* f_2\, ds_2, \tag{20}$$

that is known and for function $\varphi(s_1)$ defined on $\partial\Omega_1$ define the mapping $K:\partial\Omega_1 \to \partial\Omega_2$ by

$$K[\varphi(s_1)] = \int_{\partial\Omega_1} \frac{\partial G^*}{\partial n}\Big|_{s=s_2} \varphi(s_1) ds_1 . \tag{21}$$

We may characterize K as the linear operator from kind of a Hilbert transform operator with the kernel $\frac{\partial G^*}{\partial n}$ which maps the solution of Laplace equation in Ω with Dirichlet data φ on $\partial\Omega_1$ and homogeneous Neumann data on $\partial\Omega_2$ to its value on $\partial\Omega_2$. Therefore from (13), (19), (20), and (21), we obtain

$$T_D(f_1(s)) = \Psi(s) + K[T_D(f_3)]. \tag{22}$$

Now from invertibility f_1 and f_3, we find

$$T_D(\alpha) = \Psi\left(f_1^{-1}(\alpha)\right) + \int_{\partial\Omega_1} \frac{\partial}{\partial n} G^*\left(f_1^{-1}(\alpha), \beta\right) f_3'\left(f_3^{-1}(\beta)\right) \quad T_D(\beta) d\beta, \tag{23}$$

or

$$T_D = \Psi + K[T_D], \tag{24}$$

where $\alpha = f_1(s_2)$ and $\beta = f_3(s_1)$. To recover function T_D from (22), it would be necessary to make the assumption that f_1 and f_3 are strictly monotone functions on their domains. This requirement is typical of such recovery problems for partial differential equations that contain an unknown function of w, this implies that the existence of the coefficient D (w) and w [3, 8, 15, 16, 17]. The unicity solution $(D(w), w)$ to the inverse problem (1)-(5) may be obtained from the following theorem.

Theorem. *For any given piecewise - continuous functions $q, f, f_1, f_2,$ and f_3 such that $f_1(x_0, y_0) = f_3(x_0, y_0), f_1(x_1, y_1) = f_3(x_1, y_1), range_{\partial\Omega_1} f_3 \subset range_{\partial\Omega_2} f_1,$ the functions f_1 and f_3 are strictly monotoning, and the inverse problem* (1)-(5) *has a continuous solution on* $\overline{\Omega}$, *the solution pair* $(D(w), w)$ *of the problem* (1)-(5) *is unique.*

Proof. From (12), clearly the continuous solution $M(x, y)$ to the problem (6)-(7) is unique. Now, if (D_1, w_1) and (D_2, w_2) to be two pairs of solution of the problem (8)-(11), then by setting $D = D_1 - D_2$ and $V = V_1 - V_2$, where $V_1 = T_{D_1}(w_1)$ and $V_2 = T_{D_2}(w_2)$, in the problem (14)-(17), we obtion where G is Green' function for Laplace equation in Ω subject to Dirichlet condition on $\partial\Omega$, that is

$$\nabla^2 G(x,y;\xi,\eta) = \delta(x-\xi, y-\eta) \quad in\ \Omega,$$
$$G(x,y;\xi,\eta) = 0 \quad on\ \partial\Omega,$$

where δ is a Dirac delta function.

Now, using the transformation

$T_D(s) = \int_{s0}^{s} D(\eta)d\eta; s \geq s_0 \geq 0;$ s_0 is a constant number,

which was used by Cannon [2], Shidfar [5], and Rundell [7].

The problem (8)-(11) reduces to one with the unknown coefficient in divergence form. Note that $T_D'(s) = D(s) \geq D_0 > 0$, so that $T_D(s)$ is invertible. For any solution $w(x,y)$ of the inverse problem (8)-(11), if $w(x_0,y_0)$ is a given non-negative constant, then we define

$$V(x,y) = T_D(w(x,y)) = \int_{w(x_0,y_0)}^{w(x,y)} D(\eta)d\eta. \tag{13}$$

By this transformation $V(x,y)$ satisfies [2]

$$\nabla^2 V(x,y) = M(x,y) \quad in\Omega, \tag{14}$$

$$\frac{\partial V}{\partial n}(x,y) = f_2(s_2) \quad on\,\partial\Omega_2, \tag{15}$$

$$V(x,y) = \begin{cases} \int_{f_3(0)}^{f_3(s_1)} D(\eta)\,d\eta & on\ \partial\Omega_1, \quad (16) \\ \int_{f_1(0)}^{f_1(s_2)} D(\eta)\,d\eta & on\ \partial\Omega_2. \quad (17) \end{cases}$$

Now, we will assume that the Dirichlet boundary data on $\partial\Omega$ are compatible at the points (x_0,y_0) and (x_1,y_1), that is, $f_1(x_0,y_0) = f_3(x_0,y_0)$ and $f_1(x_1,y_1) = f_3(x_1,y_1)$, f_1 and f_3 are strictly monotone functions on the boundary $\partial\Omega_2$ and $\partial\Omega_1$, respectively, range$_{\partial\Omega_2}$ f_1 $\subset$ range$_{\bar{\Omega}}$ w, and range$_{\partial\Omega_1}$ $f_3 \subset$ range$_{\bar{\Omega}}$ w, where the ranges are not a single point, then it will be shown that the problem (14)-(17) leads to the existence and uniqueness of the coefficient D(w) and function $w(x,y)$. These ranges conditions may be guaranted by invoking the maximum principle and suitably restricting the functions M, f_1, f_2, and f_3. We also assume that the function f_2 is continuous on $\overline{\partial\Omega_2}$ and without loss of generality we may assume that the data have been normalized with $f_1(x_0,y_0) = f_3(x_0,y_0) = 0$.

Now, by substituting expression (12) in the problem (14)-(16), and using Green' second formula, we obtain

$$V(x,y) = \iint_\Omega G^* M\,d\xi\,d\eta - \int_{\partial\Omega_2} G^* f_2\,ds_2 + \int_{\partial\Omega_1} \frac{\partial G^*}{\partial n}.\left(\int_0^{f_3(s_1)} D(\eta)d\eta\right)ds_1, \tag{18}$$

where $G^*(\xi,\eta;x,y)$ is the Green' function for Laplace equation in Ω subject to Dirichlet conditions on $\partial\Omega_1$ and Neumann on $\partial\Omega_2$ [4, 12, 13].

where n denotes the unit outwords normal to the boundary $\partial\Omega_2$, f, f_1, f_2, and f_3 are given continuous functions on their domains, and D(w) is a Lipschitz continuous function satisfying $D(w) \geq D_0 > 0$, for some constant D_0, w, and D(w) are unknown functions which remain to be determined.

If D(w) is given, then the problem (1)-(4) would be a well-posed problem for the function w(x,y). For an unknown function D(w), we must therefore provide additional information, namely (5) to provide a unique solution pair (D(w),w) to the inverse problem (1)-(5).

If we determine a unique solution to the inverse problem (1)-(5), then we have obvious physical meaning, which asserts that a thin plastic plate lies on the plastic support under a load q, D(w), the bending rigidity, and w, deflection are given for any given boundary data f, f_1, f_2, f_3, and load q [9 and 11].

In many cases, the problem (1)-(5) may occur in theory of thin plate and fluid flow problems. For example, if D(w) is a constant function, and $f = f_1 = f_3 = 0$, then w(x,y) in the problem (1)-(4) will be the bending of a simply supported thin plate under a load q [9, 11, 14, 20, 21].

In the next section, we consider the inverse problem (1)-(5), and discribe some existence and uniqueness of results for the solution pair (D(w),w) satisfying (1)-(5). The coefficient D(w) will be determined in terms of q, f, f_1, f_2, and f_3. Some conclusions are given in section 3.

2-Existence and Uniqueness

By demonstrating the following result, we will identify the function D(w), when (D(w),w) is a solution to the inverse problem (1)-(5). For this purpose, we consider some methods introduced by Cannon [2], Matsuzawa [1], DuChateau [18], Shidfar [5,10], and Rundell [6,7]. Now, let us suppose M(x,y) = div (D(w)grad w), then equivalently, we have to couple systems of problems

$$\nabla^2 M(x,y) = q(x,y) \quad in\,\Omega, \tag{6}$$

$$M(x,y) = f(x,y) \quad on\,\partial\Omega, \tag{7}$$

and

$$div[D(w(x,y))grad \;\; w(x,y)] = M(x,y) \quad in\,\Omega, \tag{8}$$

$$w(x,y) = \begin{cases} f_1(x,y) & if \quad (x,y) \in \partial\Omega_2, \\ f_3(x,y) & if \quad (x,y) \in \partial\Omega_1, \end{cases} \tag{9, 10}$$

$$D(w(x,y))\frac{\partial w}{\partial n}(x,y) = f_2(x,y) \qquad on\ \partial\Omega_2. \tag{11}$$

The solution of the problem (6)-(7), following the argument [12] and using Green's second formula yields

$$M(x,y) = \iint_{\Omega} G(\xi,\eta;x,y)\,q(\xi,\eta)\,d\xi d\eta + \oint_{\partial\Omega} f\frac{\partial G}{\partial n}ds, \tag{12}$$

A Method for Solving an Inverse Biharmonic Problem

A. Shahrezaee

A.Shidfar

Department of Mathematics,
Iran University of Science and Technology

Abstract

This paper deals with the problem of determining of an unknown coefficient in an inverse elliptic boundary value problem. Using a nonconstant overspecified data, it has been shown that the solution to this inverse problem exists and is unique.

Key words

Nonconstancy overspecified condition, Existence, Uniqueness, Unknown coefficient, Bending, Inverse problem

AMS (MOS) subject classification. 35J40 and 35R30

1-Introduction

In this paper, we consider the problem of determining the unknown coefficient D(w) which depends only on the function w(x,y) in the following elliptic inverse nonlinear fourth order partial differential equation

$$\nabla^2[div(D(w)grad\ \ w)] = q(x,y)\ \ in\Omega, \tag{1}$$

where Ω is a bounded domain of R^2 with a sufficiently smooth boundary $\partial\Omega$ consisting of the union of the two arcs $\partial\Omega_1$ and $\partial\Omega_2$ with the common endpoints (x_0, y_0) and (x_1, y_1). $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is a Laplace operator, and q is given piecewise-continuous function in Ω. Let s_1 and s_2 be the arclengths along $\partial\Omega_1$ and $\partial\Omega_2$ measured from the point (x_0, y_0), respectively. On $\partial\Omega$, we assume that $w(x,y)$ satisfies the condition

$$div(D(w)grad\ \ w) = f(x,y), \tag{2}$$

on $\partial\Omega_1$

$$w(x,y) = f_3(s_1) \tag{3}$$

while on the $\partial\Omega_2$

$$w(x,y) = f_1(s_2), \tag{4}$$

$$D(w(x,y))\frac{\partial w}{\partial n}(x,y) = f_2(s_2), \tag{5}$$