Quasi-Permutation Representations for the Group GL(2,q) When Extended by a Certain Group of Order Two

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ABSTRACT

A square matrix over the complex field with non-negative integral trace is called a quasi-permutation matrix. For a given finite group G, let $c(G)$ be the minimal degree of a faithful representation of G by complex quasi-permutation matrices. Let $r(G)$ denotes the minimal degree of a faithful rational valued complex character of G. In this paper, we will calculate $c(G)$ and $r(G)$ for the group $GL(2,q)$ when extended by a certain group of order two .

KEYWORDS

General linear group, quasi-permutation representation, faithful representation.

1. INTRODUCTION

In [10], Wong defined a quasi-permutation group of degree n to be a finite group G of automorphisms of an n-dimensional complex vector space such that every element of G has non-negative integral trace. Also, Wong studied an extension to which some facts about permutation groups generalize to the quasi-permutation groups. In [3], the authors investigated further the analogy between permutation groups and quasi-permutation groups. They also worked over the rational field and found some interesting results. By a quasi-permutation matrix, we mean a square matrix over the complex field C with non-negative integral trace. For a finite group G, let $c(G)$ be the minimal degree of a faithful representation of G by complex quasi-permutation matrices and let $r(G)$ denotes the minimal degree of a faithful rational valued character of G .In [3], the case of equality has been investigated for abelian groups. In [4], the above quantities have been found for the group $GL(2,q)$. In this paper, we will apply the algorithms in [1] forthe group $H_2^2(q)$, where

$$H_2^2(q)=< GL(2,q),\theta \;\Big|\; \theta^2=1,\theta^{-1}A\theta=(A')^{-1}>.$$

2. BACKGROUND

Let $GL(2,q)$ denotes the general linear group of a vector space of dimension 2 over a field with q elements. Let $\theta: GL(2,q)\to GL(2,q)$ be the automorphism of $GL(2,q)$ given by $\theta(A)=(A')^{-1}$, where A' denotes the transpose of the matrix $A\in GL(2,q)$. In this case, one can define the split extension $GL(2,q).<\theta>$, that following the notations used in [6] is denoted by $H_2^2(q)$. Therefore we have

$$H_2^2(q)=< GL(2,q),\theta \mid \theta^2=1,\theta^{-1}A\theta=(A')^{-1}>.$$

Now let G denotes the group $GL(2,q)$ and let the split extension of G by the cyclic group $<\theta>$ of order 2 is denoted by G^+. Since $[G^+:G]=2$, we have $G^+=G\cup\theta G$, and the elements of G^+ which belong to G are called positive and those outside G are called negative elements. A conjugacy class in G^+ is called positive if it lies in G, otherwise it is called negative. We may assume that using [7], one can obtain information about conjugacy classes and complex irreducible characters of G, therefore so far as conjugacy classes of G^+ are concerned, one must pay attention to negative conjugacy classes of G^+.

By [8], all the conjugacy classes of $H_2^2(q)$ are real and if $A\in GL(2,q)$ is a real element, then we have

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$$\left|C^{(A)}_{H_2^2(q)}\right| = 2\left|C^{(A)}_{GL(2,q)}\right|$$

and if A is non-real, then

$$\left|C^{(A)}_{H_2^2(q)}\right| = \left|C^{(A)}_{GL(2,q)}\right|.$$

One can show that there is a one-to-one correspondence between the set of negative conjugacy classes of G^+ and the set of equivalence classes of invertible matrices in G.

Now we begin with a summary of facts relevant to the irreducible complex characters of $H_2^2(q)$.

As remarked earlier we have

$$H_2^2(q) = GL(2,q).<\theta>$$

and $\chi \in Irr(GL(2,q))$ is invariant under θ if and only if χ is a real valued irreducible character of $GL(2,q)$ and in this case there is $\varphi \in Irr(H_2^2(q))$ such that $\varphi\downarrow_{GL(2,q)} = \chi$ and φ is called an extension of χ. In fact, since $\frac{H_2^2(q)}{GL(2,q)} \cong Z_2$, therefore there are two extensions of χ, say φ and φ', whose valued on a negative element θg are related by $\varphi'(\theta g) = -\varphi(\theta g)$. Therefore, it is enough to find the values of one of these extensions. Now by using [9], it is easy to detect the real-valued irreducible characters of $GL(2,q)$. In [5], the character table of the groups $H_2^2(q)$ is given. Each of the irreducible characters is expressed as a linear combination of induced characters with integral coefficients.

In this paper, we use the same notations as used in [5] for irreducible characters of these groups.

TABLE 1

THE CHARACTER TABLE OF THE GROUP $H_2^2(q)$, q EVEN.

$\left\|C^x_{H_2^2(q)}\right\|$	$2q(q^2 -$	$2q$	$2(q-1)$	$2(q+1)$
χ	θJ	θI	$\theta\begin{pmatrix} 0 & 1 \\ -v^l & 0 \end{pmatrix}$	$\theta\begin{pmatrix} 0 & 1 \\ -w^m & 0 \end{pmatrix}$
χ_1^{q-1}	1	1	1	1
χ_q^{q-1}	q	0	1	-1
$\chi_{q+1}^{(i,q-1-i)}$	$q+1$	1	$\alpha^{il}+\alpha^{-il}$	0
$\chi_{q-1}^{(q-1)j}$	$q-1$	-1	0	$-(\beta^{jm}+\beta^{-jm})$

$GF(q)^* =<v>, 1 \le l \le \frac{q-2}{2}, 1 \le m \le \frac{q}{2}, 1 \le i \le \frac{q-2}{2}, 1 \le j \le \frac{q}{2}$, $GF(q^2)^* =<\sigma>, \sigma^{q-1} = w; \alpha, \beta \in C$ are primitive complex $(q-1)$th and $(q+1)$th roots of unity respectively.

TABLE 2

THE CHARACTER TABLE OF THE GROUP $H_2^2(q)$, q ODD.

$C^x_{H_2^2(q)}$	$2q(q^2-1)$	$4(q-1)$	$4(q+1)$	$4q$
χ	$\theta\begin{pmatrix} 0 & 1 \\ -l & 0 \end{pmatrix}$	$\theta\begin{pmatrix} 0 & 1 \\ l & 0 \end{pmatrix}$	$\theta\begin{pmatrix} 1 & 0 \\ 0 & -\delta \end{pmatrix}$	$\theta\begin{pmatrix} 1 & -2 \\ 2 & 0 \end{pmatrix}$
χ_1^{q-1}	1	1	1	1
$\chi_1^{\frac{q-1}{2}}$	1	$(-1)^{\frac{q-1}{2}}$	$-(-1)^{\frac{q-1}{2}}$	1
χ_q^{q-1}	q	1	-1	0
$\chi_q^{\frac{q-1}{2}}$	q	$(-1)^{\frac{q-1}{2}}$	$(-1)^{\frac{q-1}{2}}$	0
$\chi_{q+1}^{\left(q-1,\frac{q-1}{2}\right)}$	0	0	0	$\sqrt{q}$
$\chi_{q+1}^{(i,q-1-i)}$	$q+1$	$2(-1)^i$	0	1
$\chi_{q-1}^{(q-1)j}$	$q-1$	0	$-2(-1)^j$	-1

CONTINUE OF TABLE 2

$\left\|C^x_{H_2^2(q)}\right\|$	$4q$	$2(q-1)$	$2(q+1)$
χ	$\theta\begin{pmatrix} d & -2d \\ 2d & 0 \end{pmatrix}$	$\theta\begin{pmatrix} 0 & 1 \\ -v^l & 0 \end{pmatrix}$	$\theta\begin{pmatrix} 0 & 1 \\ -\omega^m & 0 \end{pmatrix}$
χ_1^{q-1}	1	1	1
$\chi_1^{\frac{q-1}{2}}$	1	$(-1)^l$	$(-1)^m$
χ_q^{q-1}	0	1	-1
$\chi_q^{\frac{q-1}{2}}$	0	$(-1)^l$	$-(-1)^m$
$\chi_{q+1}^{\left(q-1,\frac{q-1}{2}\right)}$	$-\sqrt{q}$	0	0
$\chi_{q+1}^{(i,q-1-i)}$	1	$\alpha^{il}+\alpha^{-il}$	0
$\chi_{q-1}^{(q-1)j}$	-1	0	$-(\beta^{jm}+\beta^{-jm})$

$GF(q)^* =<v>, 1 \le l \le \frac{q-3}{2}, 1 \le m \le \frac{q-1}{2}, 1 \le i \le \frac{q-3}{2}, 1 \le j \le \frac{q-1}{2}$, $GF(q^2)^* =<\sigma>, \sigma^{q-1} = \omega; \alpha, \beta \in C$ are primitive complex $(q-1)$th and $(q+1)$th roots of unity respectively and δ, d are fixed non-squares in $GF(q)$.

3. QUASI-PERMUTATION REPRESENTATION

We can see all the following statements in [1],[2].

Definition 3.1. Assume that E is a splitting field for G and that $F \subseteq E$. If $\chi, \psi \in Irr_E(G)$, we say that χ and

ψ are Galois conjugate over F if $F(\chi)=F(\psi)$ and there exists $\sigma\in Gal\left(\frac{F(\chi)}{F}\right)$ such that $\chi^{\sigma}=\psi$. It is clear that this defines an equivalence relation on $Irr_E(G)$, where $F(\chi)=<Q,\chi(g):g\in G>$.

Let η_i for $0\le i\le r$ be Galois conjugate classes of irreducible complex characters of G. For $0\le i\le r$, let φ_i be a representative of the class η_i, with $\varphi_0=1_G$. Write $\psi_i=\sum_{\chi_i\in\eta_i}\chi_i$ and $m_i=m_Q(\varphi_i)$ and $K_i=\ker\varphi_i$. We know that $K_i=\ker\psi_i$. For $I\subseteq\{0,1,2,\dots,r\}$ put $K_I=\bigcap_{i\in I}K_i$. By definition of $r(G)$ and $c(G)$ and using above notation we have:

$$r(G)=\min\{\xi(1):\xi=\sum_{i=1}^{r}n_i\psi_i,n_i\ge 0,K_I=1\ for\ I=\{i,i\neq 0,n_i>0\}\}$$

$$c(G)=\min\{\xi(1):\xi=\sum_{i=0}^{r}n_i\psi_i,n_i\ge 0,K_I=1\ for\ I=\{i,i\neq 0,n_i>0\}\}$$

where $n_0=-\min\{\xi(g)|g\in G\}$ in the case of $c(G)$.

In [1], the authors defined $d(\chi)$, $m(\chi)$ and $c(\chi)$ [See Definition 3.4]. Here, we can redefine it as follows:

Let χ be a complex character of G, such that $\ker\chi=1$. Then $\chi=\chi_1+\dots+\chi_n$ for some $\chi_i\in Irr(G)$.

Definition 3.2. Let χ be a complex character of G such that $\ker\chi=1$. Then define

$$(1)\ d(\chi)=\sum_{i=1}^{n}|\Gamma_i(\chi_i)|\chi_i(1)$$

$$(2)\ m(\chi)=\begin{cases}0 & if\ \chi=1_G\\ |\min\{\sum_{i=1}^{n}\sum_{\alpha\in\Gamma_i(\chi_i)}\chi_i^{\alpha}(g):g\in G\} & otherwise\end{cases}$$

$$(3)\ c(\chi)=\sum_{i=1}^{n}\sum_{\alpha\in\Gamma_i(\chi_i)}\chi_i^{\alpha}+m(\chi)1_G$$

So,

$r(G)=\min\{d(\chi):\ker\chi=1\}$ and

$c(G)=\min\{c(\chi)(1):\ker\chi=1\}$.

Lemma 3.3. Let $\chi\in Irr(G)$. Then

$(1)\ c(\chi)(1)\ge d(\chi)\ge\chi(1)$

$(2)\ c(\chi)(1)\le 2d(\chi)$.

Equality occurs if and only if $Z(\chi)/\ker\chi$ is of even order.

Proof. (1) follows from the definition of $c(\chi)(1)$ and $d(\chi)$. For (2) See [1 Lemma 3.13] .

Lemma 3.4. Let ε be a primitive $n-$th root of unity. Then $\varepsilon^{j}+\varepsilon^{-j}$, $1\le j\le n$ is rational if and only if

$$n=j,2j,3j,4j,6j,\frac{3}{2}j,\frac{4}{3}j,\frac{6}{5}j.$$

Proof. See [2 Corollary 3.2].

Lemma 3.5. If $\chi\in Irr(G)$, then $\ker\chi=\ker\sum_{\alpha\in\Gamma(\chi)}\chi^{\alpha}$. Moreover χ is faithful if and only if $\sum_{\alpha\in\Gamma(\chi)}\chi^{\alpha}$ is faithful.

Proof. See [1 Lemma 3.5].

Let $G=H_2^2(q)$, q be even, then G has four types of conjugacy classes, and four types of irreducible characters $\chi_1^{(q-1)},\chi_q^{q-1},\chi_{q+1}^{(i,q-1-i)},\chi_{q-1}^{(q-1)j}$.

Let $G=H_2^2(q)$, q be odd, then G has seven types of conjugacy classes, and seven types of irreducible characters

$$\chi_1^{(q-1)},\chi_1^{\frac{q-1}{2}},\chi_q^{q-1},\chi_q^{\frac{q-1}{2}},\chi_{q+1}^{(q-1,\frac{q-1}{2})}\chi_{q+1}^{(i,q-1-i)},\chi_{q-1}^{(q-1)j}.$$

Lemma 3.6. a) let $G=H_2^2(q)$, $q=p^n$, p be an even prime. Then $\left(\beta^{jm}+\beta^{-jm}\right)$ is rational if and only if $q=2$, where β is a primitive complex $(q+1)$th root of unity.

b) let $G=H_2^2(q)$, q be an odd prime. Then $\left(\beta^{jm}+\beta^{-jm}\right)$ is rational if and only if $q=3$ or 5, where β is primitive complex $(q+1)$th root of unity.

Proof. a) By Lemma 3.4 we know that $\left(\beta^{jm}+\beta^{-jm}\right)\in Q$ if and only if

$$q+1=jm,2jm,3jm,4jm,6jm,\tfrac{3}{2}jm,\tfrac{4}{3}jm,\tfrac{6}{5}jm$$

where $1\le j\le\frac{q}{2}$ and $1\le m\le\frac{q}{2}$. Now it is easy to see that $\beta^{jm}+\beta^{-jm}$ is rational if and only if $q=2$. Case (b) is proved similarly.

Theorem 3.7. Let $G=H_2^2(q)$, where $q=2,3,5$. Then

$r\left(H_2^2(2)\right)=1$, $c\left(H_2^2(2)\right)=2$

$r\left(H_2^2(3)\right)=2$, $c\left(H_2^2(3)\right)=3$

$r\left(H_2^2(5)\right)=4$, $c\left(H_2^2(5)\right)=6$

Proof. By Lemma 3.6 and Tables $(1),(2)$ we know that $\chi_{q-1}^{(q-1)j}$ is a rational values character, so $d=\left(\chi_{q-1}^{(q-1)j}\right)=q-1$ and this is the minimal values. This implies $r(G)=q-1$. Now we have

$$m\left(\chi_{q-1}^{(q-1)j}\right)=\begin{cases}1 & if\quad q=2,3\\ 2 & if\quad q=5\end{cases}$$

Therefore,

$$c\left(\chi_{q-1}^{(q-1)j}\right)(1)=\begin{cases}q & if\quad q=2,3\\ q+1 & if\quad q=5\end{cases}$$

This completes the proofs.

Theorem 3.8. Let $G=H_2^2(q)$, $q=p^n$, then

1) $r(G)=q$

2) $c(G)=q+1$

Proof. Let q be even, then by Table (1) and Lemma

3.3 we have

$$d\left(\chi_{q+1}^{(i,q-1-i)}\right) \geq q+1,\ c\left(\chi_{q+1}^{(i,q-1-i)}\right)(1) > q+1,$$

$$d\left(\chi_{q-1}^{(q-1)j}\right) = |\Gamma_j|(q-1) \geq q-1$$

where $\Gamma_j = \Gamma\left(Q\left(\chi_{q-1}^{(q-1)j} : Q\right)\right)$.

By using Lemma 3.6 if $q \neq 2$, then $|\Gamma_j| \geq 2$. So in this case

$$d\left(\chi_{q-1}^{(q-1)j}\right) \geq 2(q-1),$$

and $c\left(\chi_{q-1}^{(q-1)j}\right)(1) \geq 2q$.

The character χ_q^{q-1} is a rational values character so, $d\left(\chi_q^{q-1}\right) = q$, and by Table(2) $m\left(\chi_q^{q-1}\right) = 1$.

Now let q be odd, by Table(2) we have $d\left(\chi_q^{q-1}\right) = q = d\left(\chi_q^{\frac{q-1}{2}}\right)$, since χ_q^{q-1}, $\chi_q^{\frac{q-1}{2}}$ are rational values characters.

The character $\chi_{q+1}^{\left(q-1,\frac{q-1}{2}\right)}$ is a rational values character so by Table(2) $d\left(\chi_{q+1}^{\left(q-1,\frac{q-1}{2}\right)}\right) = q+1$ and $m\left(\chi_{q+1}^{\left(q-1,\frac{q-1}{2}\right)}\right) = \sqrt{q}$.

By Lemma 3.3 we have $d\left(\chi_{q+1}^{(i,q-1-i)}\right) \geq q+1$ and $m\left(\chi_{q+1}^{(i,q-1-i)}\right) \geq q+2$.

Finally, if $q \neq 3,5$ then by Lemma 3.6 $\chi_{q-1}^{(q-1)j}$ is not rational, so $|\Gamma| \geq 2$ where $\Gamma = \Gamma\left(Q\left(\chi_{q-1}^{(q-1)j} : Q\right)\right)$ therefore $c\left(\chi_{q-1}^{(q-1)j}\right)(1) \geq d\left(\chi_{q-1}^{(q-1)j}\right) \geq 2(q-1) \geq 2q$.

The values are set out in the following Tables.

TABLE 3 (q IS EVEN)

χ	$d(\chi)$	$c(\chi)(1)$
χ_1^{q-1}	not faithful	not faithful
χ_q^{q-1}	q	$q+1$
$\chi_{q+1}^{(i,q-1-i)}$	$\geq q+1$	$> q+1$
$\chi_{q-1}^{(q-1)j}$	$\geq 2(q-1)$	$> 2q$

TABLE 4 (q IS ODD)

χ	$d(\chi)$	$c(\chi)(1)$
χ_1^{q-1}	not faithful	not faithful
$\chi_1^{\frac{q-1}{2}}$	not faithful	not faithful
χ_q^{q-1}	q	$q+1$
$\chi_q^{\frac{q-1}{2}}$	q	$q+1$

Now by Tables (3),(4) we have :

$$\min\{d(\chi) : Ker(\chi) = 1\} = q$$

And $\min\{c(\chi) : Ker(\chi) = 1\} = q+1$.

Hence the result is follows.

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