Ultrafilters on infinite discrete semigroups and multiplicative means

M. Akbari Tootkaboni[†] and A. Riazi[‡]

ABSTRACT

In this paper by combining the concepts of ultrafilters and multiplicative means in a nice way we find a good tool for dealing with the Stone - $\breve{C}$ ech compactification of infinite discrete semigroups.

KEYWORDS

Semigroup Compactification, Multiplicative Mean, Ultrafilter, Stone - $\breve{C}$ ech Compactification.

1. INTRODUCTION

Throughout this paper $(S,.)$ will denote an infinite discrete semigroup. Let $B(S)$ be the C^*-algebra of all bounded complex-valued functions on S with supremum norm. Let $\Delta(S)$ denote the spectrum of $B(S)$. Then $\Delta(S)$ is a semigroup with multiplication defined by $\mu\nu(f) = \mu(T_\nu f)$, where

$T_\nu(f)(s) = \nu(L_s f)$ for $s \in S, \mu, \nu \in \Delta(S)$ and $f \in B(S)$. Furthermore, $\Delta(S)$ with the Gelfand topology (i.e. the relative weak*- topology inherited by $B(S)^*$) is a compact right topological semigroup such that the evaluation mapping $\varepsilon : S \to \Delta(S)$ is a continuous homomorphism with dense image and $\varepsilon(S) \subseteq \Lambda(\Delta(S)) = \{\mu \in \Delta(S) :$ $t \to \mu t : \Delta(S) \to \Delta(S)$ is continuous.$\}$. We can extend $f : S \to \mathbb{C}$ uniquely to $\hat{f} : \Delta(S) \to \mathbb{C}$ defined by $\hat{f}(\mu) = \mu(f)$ for every $\mu \in US$. On the other hand, $(\Delta(S), \varepsilon)$ is semigroup compactification of S. For more details the readers may see [1].

Let $P(S)$ be the set of all subsets of S then $\mathcal{A} \subseteq P(S)$ is called a filter if

(i) $\varnothing \notin \mathcal{A}$,

(ii) if $A, B \in \mathcal{A}$ then $A \bigcap B \in \mathcal{A}$,

(iii) if $A \in \mathcal{A}$ and $A \subseteq B$ then $B \in \mathcal{A}$.

If $\mathcal{A}$ is a filter and there is not any filter $\mathcal{B}$ such that $\mathcal{A} \subseteq \mathcal{B}$, then $\mathcal{A}$ will be called an ultrafilter.

For a discrete semigroup S, assume that $\beta S = \{p : p$ is an ultrafilter on $S\}$. For $A \subset S$ we let $\overline{A} = \{p \in \beta S : A \in p\}$.

The collection $\{\overline{A} : A \subseteq S\}$ forms a basis for a Hausdorff topology on βS. With this topology βS is the Stone-$\breve{C}$ ech compactification of S, (see[2], [3] for an extensive study of stone-$\breve{C}$ ech compactification from this point of view).

For $x \in S$, define $\hat{x} = \{A \subseteq S : x \in A\}$, and $e : S \to \beta S$ is defined by $e(x) = \hat{x}$ is continuous and embedding map. The operation "." On S extends uniquely to a multiplication + (see definition 1.1) on βS under which the pair $(\beta S, e)$ is a semigroup compactification of S.

Definition 1.1. (i) Let $\mathcal{A}$ be a filter on S. Then

[†] M. Akbari Tootkaboni is with the Department of Mathematics, Faculty of Sience, of Shahed University, Tehran, Iran (e-mail: akbari@shahed.ac.ir)

[‡] A. Riazi is with the Faculty of Mathematics and Computer Sience, Amirkabir University of Technology, Tehran, Iran (e-mail: @aut.ac.ir).

$\bar{\mathcal{A}} = \{p \in \beta S : \mathcal{A} \subseteq p\}$.

(ii) Let $\mathcal{A}, \mathcal{B}$ be filters on S. Then $\mathcal{A} + \mathcal{B} = \{A \subseteq S : \Omega_{\mathcal{B}}(A) \in \mathcal{A}\}$ where for $A \subseteq S, \Omega_{\mathcal{B}}(A) = \{x \in S : \lambda_x^{-1}(A) \in \mathcal{B}\}$.

Note that $\bar{\mathcal{A}}$ is a closed subset of βS and that all closed subsets of βS are of this from. (see [3],[4]).

Lemma 1.2. Let $\mathcal{A}, \mathcal{B}$ be filters on S. Then

(i) $\bar{\mathcal{A}} = \bigcap_{A \in \mathcal{A}} \bar{A}$ and $\mathcal{A} = \bigcap_{p \in \bar{\mathcal{A}}} p$

(ii) $\Omega_{\mathcal{A}}(A) = \bigcap_{p \in \bar{\mathcal{A}}} \Omega_p(A)$ for every $A \subseteq S$

(iii) $\mathcal{A} \subseteq \mathcal{B} \Leftrightarrow \bar{\mathcal{B}} \subseteq \bar{\mathcal{A}}$

Proof: see [4].□

2. Interaction between Ultrafilters and Multiplicative Means

In this section we show the relation between ultrafilters and multiplicative means. In this section we define $Z(f) = \{x \in S : f(x) = 0\}$ for each $f \in B(S)$.

Lemma 2.1. (i) $\mathcal{U} = \{A \subseteq S : \mu \in \overline{\varepsilon(A)}\}$ is an ultrafilter for each $\mu \in \Delta(S)$,

(ii) Let $p \in \beta S$ be an ultrafilter , then there exists some $\mu \in \Delta(S)$ such that $\bigcap_{A \in p} \overline{\varepsilon(A)} = \{\mu\}$,

(iii) Let $\mu \in \Delta(S)$, then there exists an ultrafilter $p \in \beta S$ such that $\bigcap_{A \in p} \overline{\varepsilon(A)} = \{\mu\}$.

Proof: (i) since $\overline{\varepsilon(A)} \bigcap \overline{\varepsilon(B)} = \overline{\varepsilon(A \bigcap B)}$ for each $A \subseteq S$ and $B \subseteq S$ so $\mathcal{U}$ is a filter. Then $\{\overline{\varepsilon(A)} : A \in \mathcal{U}\}$ has the finite intersection property and so there exists an ultrafilter p_μ on S such that

$\bigcap_{A \in p_\mu} \overline{\varepsilon(A)} = \{\mu\}$ *and* $\mathcal{U} \subseteq p_\mu$.

Also

$$\bigcap_{A \in p_\mu} \overline{\varepsilon(A)} \subseteq \bigcap_{A \in \mathcal{U}} \overline{\varepsilon(A)} = \{\mu\}$$

implies that $p_\mu \subseteq \mathcal{U}$. So $\mathcal{U}$ is an ultrafilter.

(ii) Since p is an ultrafilter , so $\{\overline{\varepsilon(A)} : A \in p\}$ has the intersection property in $\Delta(S)$, and so $\{\mu\} \subseteq \bigcap_{A \in p} \overline{\varepsilon(A)}$ for some $\mu \in \Delta(S)$, and we can see that $p \subseteq \{A \subseteq S : \mu \in \overline{\varepsilon(A)}\}$ .Hence $\bigcap_{A \in p} \overline{\varepsilon(A)} = \{\mu\}$.

(iii) It follows from (i). □

Let $A \subseteq S$,we define $\chi_A = 0$ if $x \in A$ otherwise $\chi_A = 1$. Suppose that $\mathcal{A}$ is an ultrafilter on S, then there exists $\mu \in \Delta(S)$ such that $\bigcap_{A \in \mathcal{A}} \overline{\varepsilon(A)} = \{\mu\}$, therefore $\mathcal{A}^\mu$ used to show an ultrafilter corresponding to $\mu \in \Delta(S)$ such that $\bigcap_{A \in \mathcal{A}} \overline{\varepsilon(A)} = \{\mu\}$. It is obvious that

$$\mathcal{A}^\mu = \{A \subseteq S : \mu \in \overline{\varepsilon(A)}\} = \{A \subseteq S : \mu(\chi_A) = 0\},$$

since $\mu(\chi_A) = 0$ if and only if $\widehat{\chi_A}(\mu) = 0$ if and only if $\mu \in \overline{\varepsilon(A)}$. Hence

$$\beta S = \{\mathcal{A}^\mu : \mu \in \Delta(S)\}.$$

Lemma 2.2. (i) $\mathcal{A}^\mu + \mathcal{A}^\nu = \mathcal{A}^{\mu\nu}$ for each $\mu, \nu \in \Delta(S)$,

(ii) Let $\mathcal{A}$ be a filter and $\mathcal{A}^\nu$ be an ultrafilter, then $\mathcal{A} + \mathcal{A}^\mu = \bigcap_{\mathcal{A}^\nu \in \bar{\mathcal{A}}} \mathcal{A}^{\nu\mu}$,

(iii) Let $\mathcal{A}$ be a filter and $\mathcal{A}^\mu$ be an ultrafilter then we have $\mathcal{A}^\mu + \mathcal{A} \subseteq \bigcap_{\mathcal{A}^\nu \in \bar{\mathcal{A}}} \mathcal{A}^{\mu\nu}$,

(iv) Let $\mathcal{A}, \mathcal{B}$ be filters, then $\mathcal{A} + \mathcal{B} \subseteq \bigcap_{\mathcal{A}^\nu \in \bar{\mathcal{B}}, \mathcal{A}^\mu \in \bar{\mathcal{A}}} \mathcal{A}^{\mu\nu}$.

Proof: (i) For a nonempty subset A of S, we have

$$\Omega_{\mathcal{A}^\nu}(A) = \{x \in S : \lambda_x^{-1}(A) \in \mathcal{A}^\nu\} = Z(T_\nu(\chi_A)).$$

Since $T_\nu \chi_A(t) = \nu(L_t \chi_A) = (\widehat{L_t \chi_A})(\nu)$ and $L_t \chi_A(x) = \chi_A(tx)$ for every $x, t \in S$ and for every $\nu \in \Delta(S)$, we conclude that $T_\nu \chi_A(t) = (\widehat{L_t \chi_A})(\nu)$ is a characteristic functin. Hence

$$\begin{aligned}\mathcal{A}^{\mu\nu} &= \{A : \mu\nu \in \overline{\varepsilon(A)}\} \\ &= \{A : \mu(T_\nu(\chi_A)) = 0\} \\ &= \{A : Z(T_\nu(\chi_A)) \in \mathcal{A}^\mu\} \\ &= \{A : \Omega_{\mathcal{A}^\nu}(A) \in \mathcal{A}^\mu\} \\ &= \mathcal{A}^\mu + \mathcal{A}^\nu.\end{aligned}$$

(ii)

$$\begin{aligned}\mathcal{A}+\mathcal{A}^{\mu} &= \{A:\Omega_{\mathcal{A}^{\mu}}(A)\in\mathcal{A}\}\\ &= \{A:Z(T_{\mu}(\chi_A))\in\mathcal{A}\}\\ &= \{A:Z(T_{\mu}(\chi_A))\in\mathcal{A}^{\nu},\forall\mathcal{A}^{\nu}\in\bar{\mathcal{A}}\}\\ &= \{A:A\in\mathcal{A}^{\mu}+\mathcal{A}^{\nu},\forall\mathcal{A}^{\nu}\in\bar{\mathcal{A}}\}\\ &= \bigcap_{\mathcal{A}^{\nu}\in\bar{\mathcal{A}}}\mathcal{A}^{\nu\mu}.\end{aligned}$$

(iii) Since

$$\begin{aligned}\Omega_A(A) &= \{x\in S: Z(L_x\chi_A)\in\mathcal{A}\}\\ &= \{x\in S: T_{\nu}(\chi_A)(x)=0,\forall\mathcal{A}^{\nu}\in\bar{\mathcal{A}}\}\\ &= \bigcap_{\mathcal{A}^{\nu}\in\bar{\mathcal{A}}} Z(T_{\nu}(f)),\end{aligned}$$

therefore

$$\begin{aligned}\mathcal{A}^{\mu}+\mathcal{A} &= \{A:\bigcap_{\mathcal{A}^{\nu}\in\bar{\mathcal{A}}} Z(T_{\nu}(\chi_A))\in\mathcal{A}^{\mu}\}\\ &\subseteq \{A:Z(T_{\nu}(\chi_A))\in\mathcal{A}^{\mu},\forall\mathcal{A}^{\nu}\in\bar{\mathcal{A}}\}\\ &= \bigcap_{\mathcal{A}^{\nu}\in\bar{\mathcal{A}}}\mathcal{A}^{\mu\nu}\end{aligned}$$

and we have $\mathcal{A}^{\mu}+\mathcal{A}\subseteq\bigcap_{\mathcal{A}^{\nu}\in\bar{\mathcal{A}}}\mathcal{A}^{\mu\nu}$.

(iv) We have

$$\begin{aligned}\mathcal{A}+\mathcal{B} &= \{A:\Omega_{\mathcal{B}}(A)\in\mathcal{A}\}\\ &= \{A:\bigcap_{\mathcal{A}^{\mu}\in\bar{\mathcal{B}}} Z(T_{\mu}\chi_A)\in\mathcal{A}\}\\ &\subseteq \{A:Z(T_{\mu}\chi_A)\in\mathcal{A},\forall\mathcal{A}^{\mu}\in\bar{\mathcal{B}}\}\\ &= \bigcap_{\mathcal{A}^{\mu}\in\bar{\mathcal{A}},\mathcal{A}^{\nu}\in\bar{\mathcal{B}}}\mathcal{A}^{\mu\nu}.\square\end{aligned}$$

Now define the function $\phi:\Delta(S)\to\beta S$

by $\phi(\mu)=\mathcal{A}^{\mu}$. ϕ is one-to-one and onto, by Lemma 2.1. Since for every $A\subseteq S$,

$$\begin{aligned}\phi^{-1}(\bar{A}) &= \{\mu\in\Delta(S):\mathcal{A}^{\mu}\in\bar{A}\}\\ &= \{\mu\in\Delta(S):\mu\in\overline{\varepsilon(A)}\}\\ &= \overline{\varepsilon(A)},\end{aligned}$$

so ϕ is continuous. By Lemma 2.2, it is obvious that ϕ is homomorphism.

Now by combining the concepts of ultrafilters and multiplicative means we prove some well-known results in a simpler way, Lemma 2.3, Lemma 2.4, Lemma 2.5, Lemma 2.7, Corollary 2.8 and Theorem 2.10 has proved in [4] and we will prove those by this methods.

Lemma 2.3. Suppose $\mathcal{A}$ be a filter on S such that $\mathcal{A}\subseteq\mathcal{A}+\mathcal{A}$, then $\bar{\mathcal{A}}$ is a subsemigroup of βS.

Proof: By Lemma 23.4, we have $\mathcal{A}\subseteq\mathcal{A}+\mathcal{A}\subseteq\bigcap_{\mathcal{A}^{\nu},\mathcal{A}^{\mu}\in\bar{\mathcal{A}}}\mathcal{A}^{\mu\nu}$. Therefore $\mathcal{A}\subseteq\mathcal{A}^{\mu}+\mathcal{A}^{\nu}$ for each $\mathcal{A}^{\mu},\mathcal{A}^{\nu}\in\bar{\mathcal{A}}$. Hence $\mathcal{A}^{\mu}+\mathcal{A}^{\nu}\in\bar{\mathcal{A}}$ for every

$\mathcal{A}^{\mu},\mathcal{A}^{\nu}\in\bar{\mathcal{A}}$. □

Lemma 2.4. Let $\mathcal{A}$ be a filter on S and $p\in\beta S$, then $\overline{\mathcal{A}+p}=\bar{\mathcal{A}}+p$.

Proof: Suppose that $p=\mathcal{A}^{\mu}$ for some $\mu\in\Delta(S)$. By Lemma 3.2, we have $\mathcal{A}+\mathcal{A}^{\mu}=\bigcap_{\mathcal{A}^{\nu}\in\bar{\mathcal{A}}}\mathcal{A}^{\nu\mu}$. Therefore by lemma 1.2.

$$\begin{aligned}\overline{\mathcal{A}+\mathcal{A}^{\mu}} &= \overline{\bigcap_{\mathcal{A}^{\nu}\in\bar{\mathcal{A}}}\mathcal{A}^{\nu\mu}}\\ &= \{\mathcal{A}^{\nu\mu}:\mathcal{A}^{\nu}\in\bar{\mathcal{A}}\}\\ &= \{\mathcal{A}^{\nu}+\mathcal{A}^{\mu}:\mathcal{A}^{\nu}\in\bar{\mathcal{A}}\}\\ &= \bar{\mathcal{A}}+\mathcal{A}^{\mu}.\ \square\end{aligned}$$

Let $\mathcal{A}$ be a filter on S, then $\bar{\mathcal{A}}$ is a left ideal if and only if there exists $\mathcal{A}^{\mu}\in\beta S$ such that $\mathcal{A}=\{S\}+\mathcal{A}^{\mu}$. Also $\bar{\mathcal{A}}$ will be a minimal left ideal if and only if μ be a minimal idempotent in $\Delta(S)$.

Lemma 2.5. Let $\mathcal{A}$ be a filter on S, then $\bar{\mathcal{A}}$ is a left ideal of βS if and only if $\Omega_A(A)=S$, for every $A\in\mathcal{A}$.

Proof: Let $\bar{\mathcal{A}}$ be a left ideal then there exists $\mathcal{A}^{\mu}$ such that $\mathcal{A}=\{S\}+\mathcal{A}^{\mu}$. So $Z(T_{\mu}(\chi_A))=S$ for each $A\in\mathcal{A}$, by Definition 1.1. Since

$$\varepsilon(x)\varepsilon(y)\mu(\chi_A)=\varepsilon(y)(T_{\mu}\chi_A)(x)=0$$

for each $x,y\in S$ and $A\in\mathcal{A}$, therefore $\varepsilon(x)\nu\mu(\chi_A)=0$ for every $x\in S,\nu\in\Delta(S)$ and $A\in\mathcal{A}$. Hence $Z(T_{\nu\mu}(\chi_A))=S$ for each $A\in\mathcal{A}$ and $\nu\in\Delta(S)$ and so

$$\begin{aligned}\Omega_A(A) &= \bigcap_{\mu\in\bar{\mathcal{A}}} Z(T_{\mu}\chi_A)\\ &= \bigcap_{\nu\in\beta S} Z(T_{\nu\mu}\chi_A)=S.\end{aligned}$$

Conversely, let $\Omega_A(A)=S$, for every $A\in\mathcal{A}$. Therefore

$$\Omega_A(A)=\bigcap_{\mu\in\bar{\mathcal{A}}} Z(T_{\mu}\chi_A)=S$$

implies that $\varepsilon(x)\mu(\chi_A)=T_{\mu}\chi_A(x)=0$ for each $x\in S$ and $\mu\in\bar{\mathcal{A}}$. Therefore $\varepsilon(x)\mu\in\bar{A}$ for every

$x \in S$ and $\mu \in \bar{\mathcal{A}}$, and so $\nu\mu \in \bar{\mathcal{A}}$ for each $\nu \in \beta S$ and $\mu \in \bar{\mathcal{A}}$. Now by Lemma 1.2, implies that $\beta S + \bar{\mathcal{A}} \subseteq \bar{\mathcal{A}}$ and so $\bar{\mathcal{A}}$ is a left ideal . □

Lemma 2.6. Let $\mathcal{A}$ be a filter on S, then $\bar{\mathcal{A}}$ is a left ideal of βS if and only if $Z(T_\mu(\chi_A)) = S$ for every $A \in \mathcal{A}$, where $\mu \in \Delta(S)$ and $\bar{\mathcal{A}} = \beta S + \mathcal{A}^\mu$ and also $\mathcal{A} = \{A \subseteq S : Z(T_\mu(\chi_A)) = S\}$.

Proof : Obvious. □

Lemma 2.7. Let $\mathcal{A}$ be a filter on S, then $\bar{\mathcal{A}}$ is a right ideal of βS if and only if $\mathcal{A} \subseteq \mathcal{A} + q$ for every $q \in \beta S$.

Proof: Let $\bar{\mathcal{A}}$ be a right ideal, $Z(f) \in \mathcal{A}$ implies

$$Z(f) \in \mathcal{A}^\mu + \mathcal{A}^\nu = \mathcal{A}^{\mu\nu} \in \bar{\mathcal{A}}$$

for every $\mathcal{A}^\nu \in \beta S$ and every $\mathcal{A}^\mu \in \bar{\mathcal{A}}$. On the other hand, for every $\mathcal{A}^\mu \in \bar{\mathcal{A}}$, $Z(T_\nu f) \in \mathcal{A}^\mu$ and by Lemma 1.2 $Z(T_\nu f) \in \bar{\mathcal{A}}$ and so $Z(f) \in \mathcal{A} + \mathcal{A}^\mu$. Conversely, if for every $q = \mathcal{A}^\nu \in \beta S, \mathcal{A} \subseteq \mathcal{A} + q = \mathcal{A} + \mathcal{A}^\nu$ then $\mathcal{A} \subseteq \bigcap_{\mathcal{A}^\mu \in \bar{\mathcal{A}}} \mathcal{A}^{\mu\nu} \subseteq \mathcal{A}^{\mu\nu} = \mathcal{A}^\mu + \mathcal{A}^\nu$. Therefore $\bar{\mathcal{A}}$ is a right ideal of βS. □

Suppose that $\mathcal{A}^\mu \in \beta S$ and $\mathcal{A} = \mathcal{A}^\mu + \{S\}$, then we have for every $\mathcal{A}^\nu \in \beta S$,

$$\begin{aligned}\mathcal{A} + \mathcal{A}^\nu &= (\mathcal{A}^\mu + \{S\}) + \mathcal{A}^\nu \\ &= \mathcal{A}^\mu + (\{S\} + \mathcal{A}^\nu) \\ &= \mathcal{A}^\mu + \bigcap_{\mathcal{A}^\eta \in \beta S} \mathcal{A}^{\eta\nu}.\end{aligned}$$

Since $\{S\} \subseteq \bigcap_{\mathcal{A}^\eta \in \beta S} \mathcal{A}^{\eta\nu}$,hence $\mathcal{A}^\mu + \{S\} \subseteq \mathcal{A}^\mu + \bigcap_{\mathcal{A}^\eta \in \beta S} \mathcal{A}^{\eta\nu}$, and so by Lemma 2.9 $\mathcal{A}^\mu + \{S\}$ is a right ideal.

Corollary 2.8. Let S be a commutative semigroup. Then for any filter $\mathcal{A}$ on S, $\bar{\mathcal{A}}$ is an ideal of βS if and only if $\mathcal{A} \subseteq \mathcal{A} + q$, for any $q \in \beta S$.

Proof : Let $\bar{\mathcal{A}}$ be an ideal of βS, it is obvious that $\mathcal{A} \subseteq \mathcal{A} + q$.

Conversely, if S is a commutative semigroup then for every $x \in S$ and for every $\mu \in \Delta(S)$ we have $\varepsilon(x)\mu = \mu\varepsilon(x)$. If $\mathcal{A}^\mu \in \bar{\mathcal{A}}$ then for every $x \in S$,

$$\begin{aligned}\mathcal{A} &\subseteq \mathcal{A} + \mathcal{A}^{\varepsilon(x)} \\ &= \bigcap_{\mathcal{A}^\mu \in \bar{\mathcal{A}}} \mathcal{A}^{\varepsilon(x)\mu} \\ &\subseteq \mathcal{A}^{\varepsilon(x)\mu} \\ &= \mathcal{A}^{\mu\varepsilon(x)}.\end{aligned}$$

Hence for every $\mathcal{A}^\nu \in \beta S$, $\mathcal{A}^{\nu\mu} \in \bar{\mathcal{A}}$, and so $\bar{\mathcal{A}}$ is a left ideal and the proof is completed . □

Let $e \in \Delta(S)$ be a minimal idempotent and $A \subseteq S$. If $T_{\nu e}(\chi_A)(s) = 0$ for every $s \in S$ and for some $\nu \in \Delta(S)$, then $\varepsilon(s)\nu e(\chi_A) = 0$ for every $s \in S$. Since $\overline{\varepsilon(S)} = \Delta(S)$ we will have $\mu e(\chi_A) = 0$ for each $\mu \in \Delta(S)$, and we have $A \in \mathcal{A}^{\mu e}$ for every $\mu \in \Delta(S)$. Therefore we have:

Theorem 2.9. Let $e \in \Delta(S)$ be a minimal idempotent, then

(i) $\mathcal{A}_\nu = \{A : Z(T_{\nu e}\chi_A) = S\}$ is a filter on S, for every $\nu \in \Delta(S)$.

(ii) $\mathcal{A}_\nu = \bigcap_{\mathcal{A}^\mu \in \beta S} \mathcal{A}^{\mu e}$

(iii) $\overline{\mathcal{A}_\nu}$ is a minimal left ideal βS.

Proof : (i) Since

$$\begin{aligned}Z(T_{\nu e}(f\bar{f} + g\bar{g})) &= Z(T_{\nu e}(f\bar{f}) + T_{\nu e}(g\bar{g})) \\ &= Z(T_{\nu e}(f))(\overline{T_{\nu e} f}) \\ &\quad \bigcap Z(T_{\nu e}(g)\overline{T_{\nu e}(g)}),\end{aligned}$$

$\varnothing \notin \mathcal{A}$ and since

$$Z(T_{\nu e}(f)) \subseteq Z(T_{\nu e}(f)T_{\nu e}(g)),$$

so $\mathcal{A}$ is a filter.

(ii) If $A \in \mathcal{A}_\nu$ then we have $Z(T_{\nu e}\chi_A) = S$. Therefore $\varepsilon(x)\nu e(\chi_A) = 0$ for every $x \in S$. So $\mu e(\chi_A) = 0$ for every $\mu \in \Delta(S)$,

and we have $A \in \bigcap_{\mathcal{A}^\mu \in \beta S} \mathcal{A}^{\mu e}$.

If $A \in \bigcap_{\mathcal{A}^\mu \in \beta S} \mathcal{A}^{\mu e}$, then $A \in \bigcap_{x \in S} \mathcal{A}^{\varepsilon(x)\nu e}$ and we have $\varepsilon(x)\nu e(\chi_A) = 0$ for every $x \in S$ and so $Z(T_{\nu e}\chi_A) = S$. Hence the proof is completed.

(iii) Obvious. □

Definition 2.10. Let X be a non-empty set. We define $[X]^{<\omega} = \{F \subseteq X$: non empty and finite$\}$.

Theorem 2.11. Let $\mathcal{A}$ be a filter on S such that

$\overline{\mathcal{A}}$ is a left ideal of βS. Then the following statements are equivalent.

(i) $\overline{\mathcal{A}}$ is a minimal left ideal of βS.

(ii) For every $A \in \mathcal{A}^c$ there exists $F \in [S]^{<w}$ such that $\bigcup_{x \in F}(\lambda_x^{-1}(A^c)) \in \mathcal{A}$.

Proof : (i)$\longrightarrow$(ii) : Suppose that $\overline{\mathcal{A}}$ is a minimal left ideal of βS and assume further that exists $A \in \mathcal{A}^c$ for each $F \in [S]^{<w}$ such that $\bigcup_{x \in F}(\lambda_x^{-1}(A^c)) \notin \mathcal{A}$. Then $\{B \bigcap (\bigcap(\lambda_x^{-1}(A))) : B \in \mathcal{A}, F \in [S]^{<w}\} = \mathcal{H}$ has finite intersection property. Therefore $\mu e \in \bigcap_{D \in \mathcal{H}} \overline{\varepsilon(D)}$ for some $e, \mu \in \Delta(S)$ and e is a minimal idempotent. Then $\lambda_x^{-1}(A) \in \mathcal{A}^{\mu e}$ for every $x \in S$, Hence we have $Z(T_{\mu e}(\chi_A)) = S$ and so by Lemma 3.9, $A \in \mathcal{A}$ and this contradicts the fact that $A \notin \mathcal{A}$.

(ii)$\longrightarrow$(i): Since $\overline{\mathcal{A}}$ is a left ideal in βS, so $\overline{\mathcal{A}} = \beta S + \mathcal{A}^{\mu}$ for some $\mu \in \beta S$. If $\overline{\mathcal{A}}$ is not a minimal left ideal of βS, then there exists a minimal idempotent $e \in \beta S$ such that $\overline{\mathcal{B}} = \beta S + \mathcal{A}^e \subseteq \overline{\mathcal{A}}$. Hence there exists $A \in \mathcal{B}$ such that $Z(T_e \chi_A) = S$ and $Z(T_\mu \chi_A) \neq S$, so $A \notin \mathcal{A}$. Then there exists an $F \in [S]^{<w}$ such that $\bigcup_{x \in F}(\lambda_x^{-1}(A^c)) \in \mathcal{A}$. Since $\mathcal{A} \subseteq \mathcal{B}$, we have $(\bigcap_{x \in F}(\lambda_x^{-1}(A)))^c \in \mathcal{B}$, but if $A \in \mathcal{B}$ then $\lambda_x^{-1}(A) \in \mathcal{B}$ for every $x \in S$, so $(\bigcap_{x \in F}(\lambda_x^{-1}(A)))^c \in \mathcal{B}$, and this is a contradiction. □

3. TRANSLATION INVARIANT ULTRAFILTERS

In this section we define almost translation invariant filters and ultrafilters. An ultrafilter p on S is called almost translation invariant if $A \in p$ implies that $\{x \in S : \lambda_x^{-1}(A) \in p\} \in p$. Since p is an ultrafilter, it is clear that p is almost translation invariant if and only if $p + p = p$, that is p is an idempotent. We will prove the following theorem in a simpler manner than Theorem 2.1 in [4].

Theorem 3.1. Let S be an infinite semigroup. If for any finite subset F of S there exists $x \in F^c$ such that $\lambda_x^{-1}(F)$ is a finite subset of S. Then there exists a non-fixed almost translation invariant ultrafilter on S.

Proof : Let $\mathcal{A} = \{A \subseteq S : A^c \text{ is finite}\}$. We define $\tilde{A} = \{x \in A : \lambda_x^{-1}(A^c) \text{ is finite}\}$. Since for every $A, B \in \mathcal{A}$, we have $\widetilde{A \bigcap B} = \tilde{A} \bigcap \tilde{B}$, therefore $\{\tilde{A} : A \in \mathcal{A}\}$ has finite intersection property. Now suppose that $\mathcal{B}$ is a filter generated by $\{\tilde{A} : A \in \mathcal{A}\}$. It is obvious that $\mathcal{A} \subseteq \mathcal{B}$. If $x \in \tilde{A}$ and $A \in \mathcal{A}$ then $\lambda_x^{-1}(A) \in \mathcal{A}$ and

$$\begin{aligned}\lambda_x^{-1}(\tilde{A}) &= \{t : xt \in \tilde{A}\} \\ &= \{t : \lambda_{xt}^{-1}(A^c) \text{ is finite and } xt \in A\} \\ &= \widetilde{\lambda_x^{-1}(A)}.\end{aligned}$$

Therefore

$$\begin{aligned}\tilde{A} &= \{x : \lambda_x^{-1}(\tilde{A}) = \widetilde{\lambda_x^{-1}(A)} \in \mathcal{B}\} \\ &\subseteq \{x : \lambda_x^{-1}(\tilde{A}) \in \mathcal{B}\}\end{aligned}$$

so that for every $A \in \mathcal{A}$, $\tilde{A} \in \mathcal{B} + \mathcal{B}$, and we will have $\mathcal{B} \subseteq \mathcal{B} + \mathcal{B}$. Now by Lemma 3.2 of [4], $\overline{\mathcal{B}}$ is a closed subsemigroup of βS, hence $\overline{\mathcal{B}}$ has an idempotent, by [1. chapter 1. Theorem 3.11] . It is obvious that every $p \in \overline{\mathcal{B}}$ is a non fixed ultrafilter.

4. REFERENCES

[1] J.F. Berglund, H.D. Junghenn, and P. Milnes, Analysis on semigroups. Function spaces Compactifications representations, Wiley,1989.

[2] L. Gilman and M. Jerison, Ring of continuous functions, Van Nostrand, Princeton, N.J.,1960.

[3] N.Hindman and D.Strauss, Algebra in the Stone - Č ech compactification Theory and Application, Springer series in Computational Mathematics 15, Walter de Gruyter. Berlin. New York, 1998.

[4] T. Papazyan, Filter and semigroup properties , semigroup Forum 41, (1990), 329-338.