Multipliers with Closed Range on Weighted Group Algebras

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ABSTRACT

Let G be a locally compact abelian group and $\mu \in M(G)$ be given. If μ is the product of an invertible and an idempotent measure then the ideal $\mu * L^1(G)$ is obviously closed in $L^1(G)$. The problem whether the converse is also true, which was raised by E. Hewitt, has been first studied by I. Glicksberg and later by B. Host and F. Parreau who completely solved it in their impressive paper. Thus $\mu * L^1(G)$ is closed in $L^1(G)$ if and only if μ is the product of an invertible and an idempotent measure in $M(G)$. In this paper we study the multipliers with closed range on the weighted group algebra $L^1(G,\omega)$. We prove among other things that, the Glicksberg-Host-Parreau theorem for group algebras remain valid for regular weighted group algebras.

KEYWORDS

Multipliers, Banach algebras, closed ideals, weighted group algebras, Hewitt property.

1. Introduction

Let G be a locally compact group with a fixed left Haar measure λ. A weight function ω on G is a strictly positive λ-measurable function on G with,.

$$\omega(xy) \le \omega(x)\omega(y), \quad (x, y \in G) \tag{1}$$

We assumed that $\omega(x) \ge 1$ for all $x \in G$. The weighted group algebra $L^1(G,\omega)$ is the set of all λ-measurable functions f such that:

$$\|f\|_{1,\omega} = \int_G |f|(x)\omega(x)d\lambda(x) < \infty. \tag{2}$$

Let also,

$$L^\infty(G,\omega) = \{f : f \quad is\ \lambda - measurable \quad and, \quad \|f\|_{\infty,\omega} = \|f/\omega\|_\infty < \infty\}. \tag{3}$$

Then $(L^1(G,\omega), \|\cdot\|_{1,\omega})$ and $(L^\infty(G,\omega), \|\cdot\|_{\infty,\omega})$ are Banach spaces and $L^\infty(G,\omega)$ is the dual of $L^1(G,\omega)$ with the pairing,

$$\langle f, g \rangle = \int_G f(x)g(x)d\lambda(x), \quad (f \in L^\infty(G,\omega) \quad, g \in L^1(G,\omega)). \tag{4}$$

By $M(G,\omega)$ we mean the set of all complex-valued, regular Borel measures μ on G such that,

$$\|\mu\|_\omega = \int_G \omega(x)d|\mu|(x) < \infty, \tag{5}$$

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and by $C_0(G,\omega)$ we mean the set of all continuous functions f, such that $f/\omega \in C_0(G)$. The Banach space $M(G,\omega)$ is the dual of $C_0(G,\omega)$ with the pairing,

$$<\mu,f>=\int_G f(x)d\mu(x), \quad (\mu\in M(G,\omega),\ f\in C_0(G,\omega)). \tag{6}$$

The product of $M(G,\omega)$ is the same as convolution in $M(G)$.

Let G be a locally compact abelian group and $\mu\in M(G)$ be given. If μ is the product of an invertible and an idempotent measure then the ideal $\mu * L^1(G)$ is obviously closed in $L^1(G)$. The problem whether the converse is also true, which was raised by E. Hewitt, has been first studied by I. Glicksberg [6] and later by B. Host and F. Parreau who completely solved it in their impressive paper [7]. Thus $\mu * L^1(G)$ is closed in $L^1(G)$ if and only if μ is the product of an invertible and an idempotent measure in $M(G)$. The abstract version of this problem for commutative regular algebras has been considered in several papers (see for example [8], [9]). Our aim in this paper is to establish the analogue of the Glicksberg-Host-Parreau (G-H-P) theorem for the weighted group algebra $L^1(G,\omega)$ of a locally compact abelian group G.

Let A be a commutative Banach algebra. We denote by $\Delta(A)$, the set of all non-zero multiplicative linear functionals on A. A is semisimple if the Gelfand transform $a \longrightarrow \hat{a}$ from A to $\Delta(A)$ is injective. The algebra A is saied to be regular, if every two disjoint subsets of $\Delta(A)$, one of which is compact and the other is closed, can be separated by elements of A. Moreover, if the ideal of A, consisting of all elements of A whose Gelfand transform has compact support in $\Delta(A)$, is norm dense in A, then we call A Tauberian. A subset F of $\Delta(A)$ is a set of uniqueness if for each $a\in A$, $\hat{a}(F)=\{0\}$ imply $a=0$. It is trivial that F is a set of uniqueness if and only if $\|a\|_F = sup\{\hat{a}(\gamma):\ \gamma\in F\}$ defines a norm on A. Let $M(A)$ be the Banach algebra of all multipliers on A. For $T\in M(A)$ let

$$\Delta(T)=\{\gamma\in\Delta(A):\ \hat{T}(\gamma)\neq 0\}, \tag{7}$$

and,

$$\delta(T)=inf\{|\hat{T}(\gamma)|:\ \gamma\in\Delta(T)\}. \tag{8}$$

For each ideal I of Banach algebra A let,

$$hull(I)=\{\gamma\in\Delta(A):\ \hat{i}(\gamma)=0,\ \ \forall i\in I\}. \tag{9}$$

A subset K of $\Delta(A)$ is called a boundary of A if the maximum modules of every function $\hat{a}(a\in A)$ is attained on K. The intersection of all closed boundaries of the algebra A is called the Shilov boundary of A and is denoted by $\partial(A)$. A uniform norm on a normed algebra A is a (non necessarily complete) submultiplicative norm $\|.\|$ on A satisfying the square property $\|x^2\|=\|x\|^2$. A has the unique uniform norm property if A admits exactly one uniform norm. A admits a uniform norm if and only if the spectral radius r_A is a uniform norm on A, if and only if A is commutative and semisimple. A has unique uniform norm property if r_A the only uniform norm on A.

2. Main results

As in [8] we say that a Banach algebra A has the Hewitt property if every multiplier T on $L^1(G)$ with closed range factors as a product of an invertible and an idempotent multiplier. Note that by Wendel's theorem [10] each multiplier on $L^1(G)$ has the form $f \longrightarrow \mu * f$ for a unique $\mu\in M(G)$. This means that,

$$\mathcal{M}(L^1(G))=M(G). \tag{10}$$

More generally,

$$\mathcal{M}(L^1(G,\omega))=M(G,\omega) \tag{11}$$

[5, Lemma 2.3]. Hence the G-H-P theorem asserts that for every abelian group G, $L^1(G)$ has the Hewitt property. Let us express an abstract form of Glicksberg-Host-Parreau theorem that was proved by A. Ulger [9, theorem 3.7]. According to [8] and [9] an open subset O of $\Delta(A)$ will be called a u-set with respect to the algebra A, if there is a constant $c>0$ such that, for each $\gamma\in O$, there is an element $a\in A$ with $\hat{a}(\gamma)=1$, $\|a\|\le c$ and the support of $\hat{a}$ is contained in the set O.

Theorem 2.1. *Let A be a commutative semisimple regular Tauberian Banach algebra with a bounded approximate identity. Suppose that for each $T\in\mathcal{M}(A)$ with closed range, $\Delta(T)$ is a u-set with respect to A and that $\mathcal{M}(A)$ can be identified in a natural way with the dual of a closed subspace of A^*A which is invariant under each T^*. Then,*

(a) A closed ideal of $\mathcal{M}(A)$ is unital if and only if it is a principal ideal.

(b) The range of $T\in\mathcal{M}(A)$ is closed in A if and only if T is the product of an invertible and an idempotent multiplier.

For a locally compact abelian group G it is well-known that $L^1(G)$ is a commutative semisimple Tauberian Banach algebra and has a bounded approximate identity. Moreover by [9, Lemma 3.3] every open subset of $\Delta(L^1(G)) = \hat{G}$ is a u-set with respect to $L^1(G)$. Therefore Glicksberg-Host-Parreau theorem follows from by theorem 2.1. But $L^1(G,\omega)$ is regular if and only if ω is non-quasi analytic, i.e.,

$$\sum_{n=1}^{\infty} \frac{\log \omega(nx)}{1+n^2} < \infty, \quad (\forall x \in G). \tag{12}$$

It is proved in [2] that $L^1(G,\omega)$ is semisimple in the case where G is abelian. In rest of this paper G is a locally compact abelian group. Contrary to the group algebras every open subset of $\Delta(L^1(G,\omega))$ is not necessarily a u-set with respect to $L^1(G,\omega)$. Because if every open subset of $\Delta(L^1(G,\omega))$ is a u-set with respect to $L^1(G,\omega)$, then obviously $L^1(G,\omega)$ must be regular. Note that for a locally compact abelian group G, $\Delta(L^1(G))$ is $\hat{G}$, the set of all characters on G, but $\Delta(L^1(G,\omega))$ may be very bigger than $\hat{G}$. By a $\omega-$bounded generalized character on G we mean a non-zero, complex valued, continuous function α on G such that,

$$|\alpha(x)| \le \omega(x), \quad (x \in G) \quad \textit{and}, \\ \alpha(x+y) = \alpha(x)\alpha(y), \quad (x,y \in G), \tag{13}$$

and by $H(G,\omega)$ we denote the set of all $\omega-$bounded generalized characters on G. $\Delta(L^1(G,\omega))$ is homeomorphic to $H(G,\omega)$ equipped with compact-open topology [3, Proposition 4.4]. In the following we prove that if $L^1(G,\omega)$ is regular, then for every $T \in \mathcal{M}(L^1(G,\omega))$ with closed range, $\Delta(T)$ is a u-set with respect to $L^1(G,\omega)$.

Lemma 2.2. Let T be a multiplier on $L^1(G,\omega)$ with closed range. Then $\delta(T) > 0$.

PROOF: Let T be a multiplier on $L^1(G,\omega)$ with closed range. Define $\tilde{T} : \frac{L^1(G,\omega)}{Ker(T)} \longrightarrow T(L^1(G,\omega))$ by $\tilde{T}(f + Ker(T)) = T(f)$. Since $T(L^1(G,\omega))$ is closed, by open mapping theorem $\tilde{T}$ is an isomorphism. Then $\tilde{T}^*$ is an isomorphism from $T(L^1(G,\omega))^*$ to $\frac{L^1(G,\omega)}{Ker(T)}^*$. As

$$T(L^1(G,\omega))^* = \frac{L^\infty(G,\omega)}{T(L^1(G,\omega))^\perp}, \\ \frac{L^1(G,\omega)}{Ker(T)}^* = Ker(T)^\perp = T^*(L^\infty(G,\omega)) \tag{14}$$

and

$$\tilde{T}^*(\gamma + T(L^1(G,\omega)^\perp) = \tilde{T}^*(\gamma + Ker(T)^*) \\ = T^*(\gamma), \quad (\gamma \in L^\infty(G,\omega)), \tag{15}$$

there exist a constant α such that,

$$\| \gamma + T(L^1(G,\omega)^\perp \| \le \alpha . \| T^*(\gamma) \|_{\infty,\omega} . \tag{16}$$

Hence if $\gamma \in H(G,\omega) - hull(T(L^1(G,\omega)))$, we have

$$\| \gamma \|_{\infty,\omega} = \| \gamma + T(L^1(G,\omega)^\perp \| \le \alpha . \| T^*(\gamma) \|_{\infty,\omega} \\ = \alpha \hat{T}(\gamma). \| \gamma \|_{\infty,\omega} \tag{17}$$

Therefore, $\delta(T) \ge 1/\alpha > 0$.

Theorem 2.3. *The following are equivalent:*

(a) $L^1(G,\omega)$ *is regular.*

(b) $\Delta(L^1(G,\omega)) = \hat{G}$.

PROOF: $(a) \Rightarrow (b)$. *Since $L^1(G,\omega)$ is semisimple by [3, Lemma 4.2] $H(G,\omega) \neq \emptyset$. Suppose that $\alpha \in H(G,\omega)$. Define $\Psi_\alpha : \hat{G} \longrightarrow \alpha.\hat{G}$ by $\Psi_\alpha(\theta) = \alpha.\theta$ (pointwise multiplication). Ψ_α clearly is continuous and one to one. The set $\Psi_\alpha(\hat{G}) = \alpha.\hat{G}$ is a closed subset of $H(G,\omega)$. If for all $f \in L^1(G,\omega)$ we define,*

$$\| f \|_\alpha = sup\{ |\hat{f}(\alpha.\theta)| : \theta \in \hat{G} \}, \tag{18}$$

$\| . \|_\alpha$ is a uniform norm on $L^1(G,\omega)$. Hence $\alpha.\hat{G}$ is a set of uniqueness for $L^1(G,\omega)$. Indeed this set is whole of $H(G,\omega)$. For this end let $\beta \in H(G,\omega) - \alpha.\hat{G}$. Then $\alpha.\hat{G}$ and $\beta.\hat{G}$ are closed set of uniqueness. But by [3, Theorem 4.1] $L^1(G,\omega)$ is regular if and only if has unique uniform norm property and by [1, Theorem 2.3] $L^1(G,\omega)$ has unique uniform norm property if and only if $\partial(L^1(G,\omega))$ is the smallest closed set of uniqueness. Hence $\partial(L^1(G,\omega)) \subseteq \alpha.\hat{G} \bigcap \beta.\hat{G} = \emptyset$. This contradiction shows that $H(G,\omega) = \alpha.\hat{G}$.

$(b) \Rightarrow (a)$. Let $\Delta(L^1(G,\omega)) = \hat{G}$. If $F \subseteq \hat{G}$ is closed and $\gamma \in \hat{G} - F$, then by regularity of $L^1(G)$, there exists an element $f \in L^1(G)$ such that $\hat{f}(\gamma) \neq 0$ and $\hat{f}(F) = \{0\}$. Choose a compact subset of G such that $\int_K f(x)\gamma(x)dx \neq 0$. Then by [4, Proposition 2.1] $f.\chi_F \in L^1(G,\omega)$ with $f.\chi_F(\gamma) \neq 0$ and $f.\chi_F(F) = \{0\}$.

Example 2.4. *Let* $\alpha \in \mathbb{R}^+$ *and* $\omega_\alpha(x) = (1+|x|)^\alpha$, $(x \in \mathbb{R})$. *Then* ω *is a non-*quasi analytic weight on $(\mathbb{R},+)$. Therefore $L^1(\mathbb{R},\omega_\alpha)$ is regular and by theorem 2.3, $\Delta(L^1(\mathbb{R},\omega_\alpha)) = \hat{\mathbb{R}} = \mathbb{R}$.

Corollary 2.5. The following are equivalent:

(a) $L^1(G,\omega)$ is regular.

(b) $\Delta(L^1(G,\omega)) = \partial(L^1(G,\omega)) = \hat{G}$.

Corollary 2.6. Let $L^1(G,\omega)$ is regular. Then for every $T \in \mathcal{M}(L^1(G,\omega))$ with closed range, $\Delta(T)$ is a u-set with respect to $L^1(G,\omega)$.

PROOF: By [9, Theorem 3.4] if $L^1(G,\omega)$ is regular and T is a multiplier on $L^1(G,\omega)$ with closed range, then $\delta(T) > 0$ if and only if $\Delta(T)$ is a u-set with respect to $L^1(G,\omega)$.

Corollary 2.7. Let $L^1(G,\omega)$ be regular. Then $L^1(G,\omega)$ has the Hewitt property.

PROOF: It follows from corollary 2.6 and theorem 2.1.

Remarks 2.8. **(a)** Let $\mu \in M(G,\omega)$ and T be define on $L^1(G,\omega)$ by $T(f) = \mu * f$. If $\delta(T) > 0$ then since $\hat{T}$ is continuous, $\Delta(T)$ is both closed and open in $\Delta(L^1(G,\omega))$. Hence in the case that $\Delta(L^1(G,\omega)) = H(G,\omega)$ is connected, either $\Delta(T) = \{0\}$, in which case T is identically null on $L^1(G,\omega)$, or $\Delta(T) = H(G,\omega)$, in which case the ideal $\mu * L^1(G,\omega)$ is a hull-less ideal (i.e. $hull(\mu * L^1(G,\omega)) = \emptyset$), hence by Wiener's Tauberian theorem $\mu * L^1(G,\omega) = L^1(G,\omega)$. So if we define S on $\mu * L^1(G,\omega) = L^1(G,\omega)$ by $S(\mu * f) = f$, then by open mapping theorem S is a multiplier on $L^1(G,\omega)$ and by [5, Lemma 2.3] there is a $\eta \in M(G,\omega)$ such that $f = \eta * \mu * f$ $(\forall f \in L^1(G,\omega))$. Therefore μ is invertible. Hence in the case that $H(G,\omega)$ is connected, $L^1(G,\omega)$ has the Hewitt property. For example in the case $(R,+)$, let ω be a weight on R. If

$$\lambda^+ = lim_{x \to +\infty} \frac{log\,\omega(x)}{x}, \quad \lambda^- = lim_{x \to -\infty} \frac{log\,\omega(x)}{x}, \qquad (19)$$

then it is easy to show that,

$$H(R,\omega) = \{\xi \in C : \lambda^- \leq Rel(\xi) \leq \lambda^+\}. \qquad (20)$$

Therefore $H(R,\omega)$ always is connected and $L^1(R,\omega)$ has the Hewitt property.

(b) By corollary 2.5 if $L^1(G,\omega)$ is regular then for every $T \in \mathcal{M}(A)$ with closed range, $\Delta(T)$ is a u-set with respect to $L^1(G,\omega)$. The converse of this statement is not necessarily true. For example for each $x \in \mathbb{R}$ let $\omega(x) = e^x$. Then for every multiplier T on $L^1(\mathbb{R},\omega)$ with closed range, $\Delta(T)$ is $\emptyset$ or $H(\mathbb{R},\omega)$ (see part a). Now since $L^1(\mathbb{R},\omega)$ has a bounded approximate identity [5], by [8] $H(\mathbb{R},\omega)$ is a u-set with respect to $L^1(\mathbb{R},\omega)$. But since ω is not non-quasi analytic, then $L^1(\mathbb{R},\omega)$ is not regular.

(c) As we mentioned before, for a Banach algebra A, if every open subset of $\Delta(A)$ is a u-set with respect to A, then A is regular. Part (b) shows that we can not replace "every open subset of $\Delta(A)$ is a u-set with respect to A " with "for every $T \in \mathcal{M}(A)$ with closed range, $\Delta(T)$ is a u-set with respect to A " in this statement.

(d) If ω is multiplicative, then $L^1(G,\omega)$ and $L^1(G)$ are isometrically isomorphic and hence $L^1(G,\omega)$ has the Hewitt property. Also if $L^1(G,\omega)$ is amenable (or equivalently G is amenable and $\omega^*(x) = \omega(x)\omega(x^{-1})$ is bounded), by [11] there is an element $\tilde{\omega} \in H(G,\omega)$ such that $L^1(G,\tilde{\omega})$ and

$L^1(G,\omega)$ are equal by equivalent norms. Hence if $L^1(G,\omega)$ is amenable then it has the Hewitt property.
The following theorem is well-known for group algebras $(\omega = 1)$ [6], and also for commutative regular Banach algebras that satisfy the hypothesis of theorem 2.1 [9, Lemma 3.3].

Theorem 2.9. Let $\mu \in M(G,\omega)$. Then the following are equivalent:

(a) $\mu * L^1(G,\omega)$ *is closed in* $L^1(G,\omega)$.

(b) $\mu * M(G,\omega)$ *is closed in* $M(G,\omega)$.

PROOF: $(a) \Rightarrow (b)$. Let T be the mapping $f \longrightarrow \mu * f$ on $L^1(G,\omega)$. Let also $\{\eta_\alpha\}_{\alpha \in I}$ be a net in $M(G,\omega)$ such that, $lim_\alpha \mu * \eta_\alpha = \theta \in M(G,\omega)$. Then if $\{e_\beta\}_{\beta \in J}$ is a bounded approximate identity for $L^1(G,\omega)$, for each $\beta \in J$, $lim_\alpha \mu * \eta_\alpha * e_\beta = \theta * e_\beta$. Since $\eta_\alpha * e_\beta \in L^1(G,\omega)$ and T has closed range, there is an $f_\beta \in L^1(G,\omega)$ such that $\theta * e_\beta = \mu * f_\beta$. Moreover since $\widetilde{T} : L^1(G,\omega)/Ker(T) \longrightarrow \mu * L^1(G,\omega)$ is a isomorphism, we can assume that $\{f_\beta\}_{\beta \in J}$ is bounded. Hence by Banach-Alaoglu theorem there is a subnet of $\{f_\beta\}_{\beta \in J}$, also denoted by $\{f_\beta\}_{\beta \in J}$, that converges in the $weak^*$-topology of $M(G,\omega)$ to an element $\vartheta \in M(G,\omega)$. Now by [5] $e_\beta \longrightarrow \delta_e$ in the $weak^*$-topology of $M(G,\omega)$, where e is the identity element of G and δ_e is the unit mass concentrated at e. Therefore,

$$\theta = weak^* - \lim_\beta \theta * e_\beta = weak^* - \lim_\beta \mu * f_\beta = \mu * \vartheta. \tag{21}$$

Hence $\mu * M(G,\omega)$ is closed in $M(G,\omega)$.

$b \Rightarrow a$. Let T be the mapping $\vartheta \longrightarrow \mu * \vartheta$ on $M(G,\omega)$. Then the mapping $\vartheta + Ker(T) \longrightarrow \mu * \vartheta$ is an isomorphism from $M(G,\omega)/Ker(T)$ to $\mu * M(G,\omega)$. Since $L^1(G,\omega)$ is an ideal in $M(G,\omega)$ and $L^1(G,\omega)$ has a bounded approximate identity,

$$\|f + Ker(T|_{L^1(G,\omega)})\| \le \|f + Ker(T)\|, \quad (f \in L^1(G,\omega)). \tag{22}$$

Hence the mapping $f + Ker(T) \longrightarrow \mu * f$ is an isomorphism from $L^1(G,\omega)/Ker(T|_{L^1(G,\omega)})$ to $\mu * L^1(G,\omega)$ and $\mu * L^1(G,\omega)$ is closed in $L^1(G,\omega)$.

Note that if $\mu * M(G,\omega)$ is closed in $M(G,\omega)$ and T is the mapping $\vartheta \longrightarrow \mu * \vartheta$ on $M(G,\omega)$, then since the multiplication in $M(G,\omega)$ is $weak^*$ continuous, $T = L^*$ for some continuous linear operator $L : C_0(G,\omega) \longrightarrow C_0(G,\omega)$. Indeed since

$$< \mu * \vartheta, g > = < T(\vartheta), g > = < L^*(\vartheta), g > = < \vartheta, L(g) >, \quad (g \in C_0(G,\omega)), \tag{23}$$

it is easy to see that for all $g \in C_0(G,\omega)$ and $x \in G$,

$$L(g)(x) = \int_G g(yx)\, d\mu(y). \tag{24}$$

Moreover since an operator on Banach spaces has closed range if and only if its adjoint has, the two statements in theorem 2.9 are equivalent to the fact that $L(C_0(G,\omega))$ is closed in $C_0(G,\omega)$. The following proposition follows from theorem 2.9 and [9, Lemma 2.1, Theorems 2.4, 2.5 and Proposition 2.3].

Proposition 2.10. *Let* T *be the mapping* $\vartheta \longrightarrow \mu * \vartheta$ *on* $M(G,\omega)$ *and* L *be as above. Then if* T *has closed range, the following are equivalent:*

(a) μ *is the convolution of an invertible and an idempotent measure.*

(b) $\mu * M(G,\omega)$ *has a unit element.*

(c) $\mu * L^1(G,\omega)$ *has a bounded approximate identity.*

(d) $\mu * M(G,\omega) + Ker(T)$ *is dense in* $M(G,\omega)$.

(e) $\mu * \mu * M(G,\omega)$ *is dens in* $\mu * M(G,\omega)$.

(f) $\mu * \mu * L^1(G,\omega)$ *is dens in* $\mu * L^1(G,\omega)$.

(g) $Ker(L^2) = Ker(L)$.

(h) $\delta(T) > 0$.

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