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$$\inf\{\|\Pi(\mu)x\| : \mu \in M_o(G)\} = \text{dis}(x, K_{H,\Pi}) \text{ for all } x \in H$$

Proof. Let $\{\mu_a\}$ be a net in $M_o(G)$ which satisfies $\|\mu_\alpha * \mu * \mu\| \to 0$ for any $\mu \in M_o(S)$. For any $y \in H$ and $\mu \in M_o(G)$ we have

$$\|\Pi(\mu_\alpha)(y - \Pi(\mu)y\| = \|\Pi(\mu_\alpha)y - \Pi(\mu_\alpha * \mu)y\|$$

$$= \|\Pi(\mu_\alpha - \mu_\alpha * \mu)y\|$$

$$\leq \|(\mu_\alpha - \mu_\alpha * \mu)\| \|y\| \to 0$$

hence by linearity of $\Pi(\mu_\alpha)$ we conclude that $\|\Pi(\mu_\alpha)z\| \to 0$ fo all $z \in K_{H,\Pi}$.

Now given $\varepsilon > 0$ there is $z \in K_{H,\Pi}$ such that $\|x+z\| \leq \text{dis}(x, K_{H,\Pi}) + \varepsilon$ and since $\|\Pi(\mu_\alpha)z\| \to 0$, there is α_o such that $\|\Pi(\mu_{\alpha o})z\| < \varepsilon$. Hence

$$\|\Pi(\mu_{\alpha o})x\| \leq \|\Pi\mu_{\alpha o}(x+z)\| + \|\Pi(\mu_{\alpha o})z\|$$

$$\leq \|x+z\| + \varepsilon$$

$$< \text{dis}(x, K_{H,\Pi}) + 2\varepsilon$$

Thus $\inf\{\|\Pi(\mu)x\|; \mu \in M_o(G)\} \leq \text{dis}(x, K_{H,\Pi})$. clearly

$\{\Pi(\mu)x : \mu \in M_o(G)\} \subseteq x + K_{H,\Pi}$ for all $x \in H$,

Thus $\inf\{\|\Pi(\mu)x\| : \mu \in M_o(G) \geq \text{dis}(0, x + K_{H,\Pi}) = \text{dis}(x, K_{H,\Pi})$ hence the result follows.

Corollary 3.2. Let G be a locally compact topologically amenable group and $\{\Pi, H\}$ be a representation of M (G), then the closure of $K_{H,\Pi}$ agrees with the set of all $x \in H$ satisfying $\inf_{\mu \in M_o(G)} \|\Pi(\mu)x\| = 0$

Corollary 3.3. Suppose G is a locally compact topologically amenable group. If $\Pi: M(G) \to B(H)$ is a faithfull representation, then $K_{H,\Pi}$ is dense in H or Π is reducible.

Proof. If G is trivial group, then M (G) is the linear span of ε_e (ε_e is the Dirac measure at the identity of G).

Without loss of generality we may assume that $\Pi(\varepsilon_e) = I$ the identity operator of B (H), (see [7]). It is easy to show that any nontrivial closed subspace of H is invariant under Π, so Π is reducible in this case.

To prove the theorem for the case that G is nontrivial, we observe that since Π is faithfull $\overline{K}_{H,\Pi} \neq \{0\}$. In fact let $\mu \in M_o(G)$ be such that $\mu \neq \varepsilon_e$ then $\Pi(\mu) \neq I$, that is there is $x \in H$ such that $\Pi(\mu)x \neq x$ or $\Pi(\mu)x - x \neq 0$, so $0 \neq \Pi(\mu)x - x \in K_{H,\Pi}$. If $K_{H,\Pi}$ is not dense in H then $\overline{K}_{H,\Pi}$ is a nontrivial closed invariant subspace of H, so Π is reducible and the proof of the theorem is complete.

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the set of all $H \in M(S)^*$ of the form $H = \sum_{i=1}^{n} F_i \times (G_i - \mu_i \Theta G_i)$ for some $F_1, ..., F_n$, $G_1, ..., G_n \in M(S)^*$ and $\mu_1, ..., \mu_2$ in $M_o(S)$. It is clear that $\mathcal{H}$ is a linear subspace of $M(S)^*$. For each $H \in \mathcal{H}$ let the orbit of H $\overline{O(H)} = \{\mu \text{ O } H: \mu \in M_o(S)\}$ and $\overline{O(H)}$ be its closure in the norm topology (which is of course the same as its weak closure, since O (H) is convex).

Theorem 2.1. The following conditions are equivalent

(a) $M(S)^*$ MTLIM.

(b) $0 \in \cap \overline{O(H)}$ where the intersection is taken over all $H \in H$.

(c) $\sup\{H(\mu): \mu \in M_o(S)\} \geq 0$ for all $H \in \mathcal{H}$.

(d) $\inf\{\|1 - H\| : H \in \mathcal{H}\} = 1$.

Proof . (a) $\Rightarrow$ (b) Suppose $M(S)^*$ has a MTLIM, then by [8, 2. 2. 1] there exists a net $\{\mu_a\}$ in $M_o(S)$ such that $\|\mu * \mu_\alpha - \mu_\alpha\| \to 0$ for any μ in $M_o(S)$.

Then for any F, G in $M(S)^*$ and μ in $M_o(S)$ we have

$$\|\mu_\alpha \Theta (F \times (G - \mu \Theta G))\| \leq \|\mu_\alpha \Theta (F \times G) - (\mu * \mu_\alpha) \Theta (F \times G)\|$$

$$\leq \|\mu_\alpha - \mu * \mu_\alpha\| \, \|F \times G\| \to 0$$

by linearity of l_μ, we conclude that $\|\mu_\alpha \Theta H\| \to 0$, for all $H \in \mathcal{H}$, hence (b). (b)$\Rightarrow$(c).

Suppose $0 \in \overline{O(H)}$ for all $H \in \mathcal{H}$ then for any $H \in \mathcal{H}$ there is a sequence $\{\mu_n\}$ in $M_o(S)$ such that $\|\mu_n \Theta H\| \to 0$. Hence

$$\|\mu_n \Theta H\| = \sup\{|H(\mu_n * \nu)| : \nu \in M(S), \|\nu\| \leq 1\}$$

$$\geq \sup\{|H(\mu_n * \nu)| : \nu \in M_o(S)\}$$

$$\geq \inf\{|H(\mu_n * \nu)| : \nu \in M_o(S)\}$$

$$\geq \inf\{|H(\theta)| : \theta \in M_o(S)\}$$

$$\geq \inf\{H(\theta) : \theta \in M_o(S)\}$$

so $\inf\{H(\theta): \theta \in M_o(S)\} \leq 0$ and since $\mathcal{H}$ is a sub space of $M(S)^*$ we conclude that $\sup\{H(\theta): \theta \in M_o(S)\} \geq 0$ for all $H \in \mathcal{H}$.

(c) $\Rightarrow$ (d). The assumption implies that $\|1 + H\| \geq 1$ for all $H \in \mathcal{H}$. If not there is some $H_o \in \mathcal{H}$ such that $\|1 + H\| = \varepsilon < 1$, in particular $1 + H_o(\mu) \leq \varepsilon$ for all $\mu \in M_o(S)$. Hence $\sup\{H_o(\mu) : \mu \in M_o(S)\} \leq \varepsilon - 1 < 0$ which is a contradiction. Since $\mathcal{H}$ is a sub-space of $M(S)^*$ we conclude that $\|1 - H\| \geq \varepsilon$ for all $H \in \mathcal{H}$. Therefore $\inf\{\|1 - H\|: H \in \mathcal{H}\} \geq 1$, since $\|1\| = 1$, hence (d) follows.

(d) $\Rightarrow$ (a). Since by our assumption $\mathcal{H}$ is not dense in $M(S)^*$, using an argument similar to [3, Lemma 3 (d) $\Leftrightarrow$ (a)] one conclude that $M(S)^*$ has a MTLIM.

Corollary 2.2. Suppose for any $\nu, \eta \in M_o(S)$ there exists λ in $M_o(S)$ such that $\nu * \lambda = \eta * \lambda$, then $M(S)^*$ has a MTLIM.

Proof. Let $H = \sum_{i=1}^{n} F_i \times (G_i - \mu_i \Theta G_i)$ for some $F_1, ..., F_n, G_1, ..., G_n$ in $M(S)^*$ and $\mu_1, ..., \mu_n$ in $M_o(S)$. Let θ be an arbitrary but fixed element of $M_o(S)$. By assumption there exists $\theta_1 \in M_o(S)$ such that $\theta * \theta_1 = (\mu_1 * \theta) * \theta_1$. Inductively suppose we have chosen $\theta_1, ..., \theta_{n-1}$, then choose $\theta_n \in M_o(S)$ such that $(\theta * \theta_1 ... * \theta_{n-1}) * \theta_n = (\mu_n * \theta * ... \theta_{n-1}) * \theta_n$ Now $\theta * \theta_1 * ... * \theta_n \in M_o(S)$ and it is easy to see that $(\theta * \theta_1 ... * \theta_n) \text{ O } H = \overline{0}$ i. e. zero is in O (H) and a fortiori in O(H) for all $H \in H$, hence by Theorem 2.1, $M(S)^*$ has a MTLIM.

3. Amenability and Representation of Locally Compact Groups

Theorem 3.1. Let G be a locally compact topologically amenable group and $\{\Pi, H\}$ a representation of M (G), then

1. Extremely Amenable Semigroups

Let S be a discrete semigroup and A be a uniformally colsed left invariant subalgebra of m (S). Denote by P_A the set of all h $\in$ m (S) of the form $h = |g - l_s g|$, for some g $\in$ A, s $\in$ S. Also let H_A be set of all h $\in$ A which have a representation of the form $h = \sum_j^n f_j (g_j - l_{s_j} g_j)$ for some $f_j, g_i \in A$, $s_j \in S$, $1 \leq j \leq n$. In case A = m (S) we denote P_A by P. If m (S) is ELA we say that S is ELA.

First we offer a Lemma.

Lemma 1.1. Let A be a uniformally closed subalgebra of m (S) then

(i) A is a lattice. If in addition A is left invariant then $P_A \subseteq A$

(ii) If f $\in$ A and f $\geq$ 0 then $\sqrt{f} \in A$.

(iii) $|m (fg))|^2 \leq m (f^2) m (g^2)$ for every mean m on A and all f, g $\in$ A.

(iv) m (|f|) = 0 implies that m (f) = 0 for every mean m on A and f $\in$ A.

Proof. (i) That A is a lattice is known by [9], hence if in addition A is left invariant, then $P_A \subseteq A$.

(ii) Let m_c (S) be the space of bounded complex valued function on S with supremum norm. With conjugation as involution, m_c (S) is a C^* - algebra. Now A + iA is a closed subalgebra of m_c (S). If we consider f as an element of the C^* - algbra A + iA, it is easy to see that the spectrem of f is contained in $[0, \infty)$, in fact if $\lambda \notin [0, \infty)$ then

$$\frac{1}{|f - \lambda|} \leq \frac{1}{\text{Im}\lambda} \quad \text{if} \quad \text{Im}\lambda \neq 0$$

$$\frac{1}{|f - \lambda|} \leq -\frac{1}{\lambda} \quad \text{if} \quad \text{Im}\lambda = 0$$

so by [1, Proposition 3.5] $f = g^2$ for some self-adjoint, hence real-valued function g. Therefore $\sqrt{f} = g \in A$.

(iii) Similar to the proof of Cauchy - Schwartz inequality.

(iv) If m $\bar{f}$ = 0 then $m (f^+ + f^-) = 0$, so $m (f^+) = m (f^-) = 0$ i. e. m (f) = 0

Theorem 1.2. Let A be a uniformly closed left invariant subalgebra of m (S) with 1 $\in$ A. Then A is ELA if and only if there is a mean m$\in A^*$ such that $m(P_A)=\{0\}$.

Proof. Suppose A is ELA and m be a multiplicative left invariant mean on A, then $m (f - l_s f)^2 = 0$ for all f $\in$ A, s $\in$ S. So by Lemma 1.1, with f replaced by $|f - l_s f|$ and g replaced by 1, we obtain

$$(m (|f - l_s f |))^2 \leq m (f - l_s f)^2 = 0$$

hence $m (P_A) = \{0\}$.

Conversely, suppose there is mean m $\in A^*$ such that $m (P_A) = \{0\}$. By parts (i) and (ii) of Lemma 1.1 we have $|g - l_s g|^{1/2} \in A$, for all g $\in$ A, s $\in$ S, therefore by Lemma 1.1 (iii) we have

$$|m (|g - l_s g|^{1/2} |g - l_s g|^{3/2})|^2 \leq m (|g - l_s g|) m (|g - l_s g|^3) = 0$$

hence $m (g - l_s g)^2 = 0$.

Now another application of Lemma 1.1 (iii) shows that

$$m(|f (g - l_s g)|^2 \leq m(f^2) m (g - l_s g)^2 = 0$$

for all f $\in$ A. Hence by Lemma 1.1 (iv), $m (f (g - l_s g)) = 0$ i. e. $m (H_A) = \{0\}$, therefore H_A is not dense in A, so by [6, Lemma 3], A is ELA.

Corollary 1.3. S in ELA if and only if there is a mean m on m(S) such that m (P) = 0.

2. Extremely Amenable Locally Compact Semigroups

In analogy to discrete case let $\mathcal{H}$ denotes

$M(S)^*$ via the identification $M(S) = C_0(S)^*$. For F, G in $M(S)^*$ we denote the multiplication of F and G by $F \times G$. In [8] it is shown that $F \times G$ is defined via the following three steps.

(i) For any $\mu \in M(S)$ and $f \in C_0(S)$, $\mu_f \in M(S)$ is defined by

$$\int g d\mu_f = \int g f d\mu \text{ for all } g \in C_0(S)$$

(ii) For any $\mu \in M(S)$ and $G \in M(S)^*$, $G \times \mu \in M(S)$ is defined by

$$\int f d(G \times \mu) = G(\mu_f) \text{ for all } f \in C_0(S)$$

(iii) For any $F, G \in M(S)^*$, $F \times G \in M(S)^*$ is defined by

$$(F \times G)(\mu) = F(G \times \mu) \text{ for all } \mu \in M(S)$$

then $M(S)^*$ becomes a commutative Banach algebra with identity [8, Theorem 1.23].

A TLIM, M on $M(S)^*$ is called a multiplicative topological left invariant mean (MTLIM) if

$$M(F \times G) = M(F) M(G) \text{ for all } F, G \in M(S)^*$$

Let G be a locally compact group and let H be a Hillbert space and $\{\Pi, H\}$ be a representation of M (G) (See [1] for definition of representation). It is known that Π is continuous, in fact $\|\Pi(\mu)\| \leq \|\mu\|$ for all $\mu \in M(G)$ (See [7]). A subspace K of H is said to be invariant under Π if $\Pi(\mu) K \subseteq K$ for all $\mu \in M(G)$. Π is called irreducible if $\{0\}$ and H are the only closed invariant subspaces of H. We say that Π is faithfull if Π is one-to-one. Let $H = L^2(G)$ and consider $\Pi: M(G) \rightarrow B(L^2(G))$ defined by $\Pi(\mu) f = \mu * f$, where $(\mu * f)(x) = \int f(y^{-1}x) d\mu(y)$ then Π is called regular representation of M (G). The regular representation is faithfull [7].

Let $\{\Pi, H\}$ be a representation of M (G), then $K_{H,\Pi}$ will denote the linear span of $\{y - \Pi(\mu) y : y \in H, \mu \in M_0(G)\}$. For $x \in H$, let $dis(x, K_{H,\Pi})$ denote the distance of x from $K_{H,\Pi}$.

Now let S be a discrete semigroup and m (S) be the Banach algebra of all bounded real valued functions on S with supremum norm. If $f \in m(S)$ and $s \in S$, let $l_s f(x) = f(sx)$ for any $x \in S$.

Let A m (S) be a uniformly colsed left invariant (i. e $l_s f \in A$ for any $f \in A$ and $s \in S$) subalgebra of m (S) with $1 \in A$ (1 is the constant one function on S). A linear functional $m \in A^*$ (the continuous dual of A) is a mean if $\varphi(f) \geq 0$ for any $f \geq 0$, $f \in A$ and $\varphi(1) = 1$, this is equivalent to the condition that.

$$\inf\{f(x) : x \in S\} \leq m(f) \leq \sup\{f(x) : \in S\}$$

for all $f \in A$.

We say that the subalgebra A is extremely left amenable (ELA) if there is a multiplicative left invariant mean on A, i.e. a mean m on A such that $m(l_s f) = m(f)$ and $m(fg) = m(f) m(g)$ for all $f, g \in m(S)$ and all $s \in S$.

Extremely left amenable semigroups were introduced for the first time by T. Mitchell [9] and later on studied by E. Granirer [4], [5], [6], and J. C. S. Wong [14] and for topological case by J. M. Ling [8] and A. Riazi [10] and [12].

Invited Paper

Three Results on Amenability

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Abstract

In this article we offer three results concerning amenability of discrete semigroups, topological semigroups and topological groups.

Introduction

Let S be a locally compact topological semigroup with measure algebra M (S). Let $M_o(S) = \{\mu \in M(S) : \mu \geq 0 \text{ and } \|\mu\| = 1\}$ be the set of all probability measures in M(S), $M_o(S)$ is a semigroup with convolution as multiplication.

For each $\mu \in M(S)$ we denote the operator $l_\mu: M(S)^* \rightarrow M(S)^*$ where $(l_\mu F)(v) = F(\mu * v)$, $v \in M(S)$ by $\mu \, O \, F$. Also denote by 1 the element in $M(S)^*$ such that $1(\mu) = \mu(S)$ for all $\mu \in M(S)$. An element $M \in M(S)^{**}$ is called a mean if

$$\inf\{F(\mu): \mu \in M_0(S)\} \leq M(F) \leq \sup\{F(\mu): \mu \in M_0(S)\} \quad (1)$$

For any $F \in M(S)^*$. Condition (1) is equivalent to

$$M(1) = \|M\| = 1 \quad (2)$$

or

$M(F) \geq 0$ for all $F \in M(S)^*$ with $F \geq 0$ and $M(1) = 1$

See [2].

A mean M is called topological left invariant (TLIM) if $M(\mu \, O \, F) = M(F)$, for any $F \in M(S)^*$ and $\mu \in M_o(S)$. If there is a topological left invariant mean on $M(S)^*$ we say that S is topological left amenable (TLA). Let $C_o(S)$ be the subalgebra of C B (S) (continuous bounded functions on S) consisting of functions which vanish at infinity. It is known that $M(S) = C_o(S)^*$ via the correspondence $\mu \rightarrow \bar{\mu}$ where $\bar{\mu}(f) = \int f d\bar{\mu}$ for any f in $C_o(S)$, [7, 14]. Under pointwise operations and supremum norm, $C_o(S)$ becomes a Banach algebra. Arens product can thus be defined in $C_o(S)^{**}$. In particular we have the following defining formulas for any f, g in $C_o(S)$, m in $C_o(S)^*$ and θ, φ in $C_o(S)^{**}$

$$(m \Theta f)(g) = m(fg)$$

$$(\varphi \Theta m)(f) = \varphi(m \Theta f)$$

$$(\theta \Theta \varphi)(m) = \theta(\varphi \Theta m)$$

this product induces a multiplication in