$= max \{k_1, k_2\}$.

since $(m,n)=1$, there exist two positive integers r and s such that $rn = ms + 1$. But from $[[a,b]_k,b^n], = 0 = [[a,b]_k, b^m]$, we get $[[a,b]_k, b^{sm}] = 0 = [[a,b]_k,b^{rn}]$; hence $b^{sm} [a,b]_{k+1} = 0$. Now considering Lemma 5 we may (and shall) assume that R is a local ring. If $b \in N$ then $[a,b] = 0$ by (i). Therefore suppose that $b \notin N$ then b^{-1} exists, and $b^{sm}[a,b]_{k+1}=0$ implies that $[a,b]_{k+1} = 0$. This completes the proof.

REMARK

The ring of quaternions shows that condition (ii) in (1), (2), (3), and (4) is essential. The non commutative ring

$$R = \left\{ \begin{bmatrix} a & b & c \\ 0 & a^2 & 0 \\ 0 & 0 & a \end{bmatrix} : a,b,c \in GF(4) \right\}$$

represented in [1] satisfies (i), (ii), and $(I)_6$; but does not satisfy Q(6). That is , Q(n) can not be dropped in (2). The non commutative ring R in [5, Remark] satisfying (i), (ii), and $(I)'_3$ shows that $(I)_n$ in (3) can not be replaced by $(I)'_n$.

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$a \in N$ and $b \in R$ $(ab+b)^m-(ba+b)^m = [a, b^m]$ for all positive integers m.

PROOF. Let $a \in N$, $b \in R$. Since N is a commutative ideal, it is easy to see that for all integers $t>1$, a^t and $(ab)^t=(ba)^t$ are both in the center of R. On the other hand for all the positive integers m we have:

$$\begin{aligned}(ab+b)^m &= (ab)^m + \binom{m}{1}(ab)^{m-1}b \\ &+\dots+ \binom{m}{m-2}(ab)^2b^{m-2} + (ab)b^{m-1} \\ &+ b(ab)b^{m-2} +\dots+ b^{m-1}(ab)b + b^m \\ &= (ba)^m + \binom{m}{1}(ba)^{m-1}b +\dots \\ &+ \binom{m}{m-2}(ba)^2b^{m-2} + ab^m + bab^{m-1} \\ &+\dots+ b^{m-1}ab + b^m\end{aligned}$$

Also,

$$(ba+b)^m = (ba)^m + \binom{m}{1}(ba)^{m-1}b +\dots+ \binom{m}{m-2}(ba)^2b^{m-2} + bab^{m-1} +\dots+ b^{m-1}ab + b^ma + b^m$$

Thus $(ab + b)^m - (ba+b)^m = ab^m - b^ma = [a,b^m]$.

LEMMA 8

If N is a commutative ideal in R and $1 \in R$, then $(I)_n$ implies that for each $a \in N$, $b \in R$ there exists a positive integer $k=k(a,b)$ such that $n[a,b]_k = 0$.

PROOF. Let $a \in N$, $b \in R$. By $(I)_n$, $(1+a)^n-1 \in Z(k)$ or $a \in Z(k)$ for some positive integer k. But N is a commutative ideal, hence for each positive integer $t>1$, $a^t \in Z \subseteq Z(k)$. Therefore in any case $na \in Z(k)$, *i.e.* $n[a,b]_k = [na,b]_k = 0$.

LEMMA 9

If N is a commutative ideal in R and R satisfies $(I)'_n$ then for each $a \in N$, $b \in R$ there exists a positive integer $k=k(a,b)>1$ such that $[[a,b]_{k-1}, b^n] = 0$.

PROOF. Let $a \in N$, $b \in R$. By $(I)'_n$,$[a,b] = 0$ or for some integer $k>1$, $(a+b)^n-b^n \in Z(K)$. But $[(a+b)^n-b^n,a+b] = -[b^n, a+b] = [a,b^n]$, thus $(a+b)^n-b^n \in Z(k)$ implies that $[[a,b^n], b]_{k-1}=0$,*i.e.* $[[a,b]_{k-1},b^n] = 0$.

Now we are ready to prove our Theorem.

PROOF OF THE MAIN THEOREM

(1) see Lemma 5 and [3, Theorem 1(5)].

(2) In view of Lemma 6, it suffices to show that if R is a local ring satisfying (i) , (ii) , $(I)_n$ and Q(n) then it is commutative. For any $a \in N$, $b \in R$ by Lemmal and Lemma 8, there exists a positive integer k such that $n[a,b]_k = 0$. Thus by Q(n),$[a,b]_k = 0$ hence R is commutative, by (1).

(3). By Lemma 6, we may assume that R is a local ring satisfying (i), (ii) and $(I)_n$ for a prime number n.

Now, if N is contained in Z then by a well known theorem of Herstein [2], (ii) impliess that R is commutative. Suppose that N is not contained in Z then, by lemma 4, there exists a prime number p such that Char. $R = p^\alpha$, $\frac{R}{N} = GF(r)$ where $r=p^\beta$.

Let $a \in N$, $b \in R$. Using (1) it suffices to show that there exists a positive integer k such that $[a,b]_k=0$. To prove this we consider the following two cases:

Case(1) $n \neq p$ (recall that p is a prime number). By Lemma 8 there exists a positive integer k such that $n[a,b]_k = 0$. But n and P are relatively prime, and Char.$R=p^\alpha$; hence $[a,b]_k = 0$.

Case (2) n=p. Let $c=(b)^{p^\beta-1}$ then $c^p=b^r$ where $r=p^\beta$ and $\frac{R}{N} = GF(r)$ hence $b-b^r \in N$,*i.e.* $[a,c^p] = [a,b^r] = [a,b]$since by (i) N is commutative. But by $(I)_n$,there exists an integer $k>1$ such that $c \in Z(k)$ or $(a+c)^n - c^n \in Z(k)$. Moreover , $[(a+c)^n - c^n, a+c] = [a,c^n] = [a,c^p] = [a,b]$; hence in any case $[a,b]_k=0$. This completes the proof.

(4). Let $a \in N$, $b \in R$. Using (1) it suffices to show that there exists a positive integer k such that $[a,b]_k=0$. But by Lemma 9, there exists. a positive integer k_1 such that

$[[a,b]_{k_1} b^n] = 0$

Also, considering Lemma7, $(II)_m$ implies that :

$[[a,b]_k , b^m] = [[a,b^m],b]_{k_2} = 0$

for some positive integers k_2 and m with $(m,n) = 1$. Therefore $[[a,b]_k , b^m] = 0 = [[a,b]_k , b^n]$, for k

(iii) For each $a \in N$, $b \in R$ there exists a positive integer $k = k(a,b)$ such that $[a,b]_k = 0$.

THEOREM

Let R be a ring which satisfies (i) and (ii). Then under any of the following additional conditions R is a subdirect sum of nil commutative and local commutative rings.

(1) R satisfies (iii).
(2) R satisfies $(I)_n$ and Q(n).
(3) R satisfies $(I)_n$ where n is a prime number.
(4) R satisfies $(II)_m$ and $(I)'_n$.

In preparation for proving the above Theorem, we establish the following Lemmas:

LEMMA 1

If R satisfies (i) and (ii), then N is a commutative nil ideal containing the commutator ideal of R.

PROOF

This is a well- known result (e.g.see[3,Lemmal(6)]).

LEMMA 2

If R satisfies (iii) or $(I)'_n$ then R is a normal ring.

PROOF

Let e be an idempotent element and $x \in R$. Then $(exe-xe)^2 = 0$, i.e. $(exe-xe) \in N$.

First suppose that R satisfies (iii), then there exists a positive integer k such that $[exe-xe, e]_k = 0$. An easy induction on k shows that $[exe-xe, e]_k = exe - xe$. Thus $exe = xe$; hence e is a centeral element in R.

Next if R satisfies $(I)'_n$ then $(exe-xe + e)-e \in N$ implies that either $(exe-xe+e)^n - e^n \in Z(k)$ for some positive integer k, or $[exe-xe,e]=0$. Therefore in any case $[exe - xe, e]_k = [(exe - xe + e)^n - e^n, e]_k = 0$. The rest of the proof proceeds as above.

LEMMA 3

If R satisfies $(I)_n$ then R is normal.

PROOF. The proof is quite similar to that of Lemma2.

LEMMA 4. Let R be a normal subdirectly irreducible ring. if R satisfies (i), (ii), and N is not contained in Z, then R is of characteristic p^{α}, where p is a prime number.

PROOF. See [3,Lemma 1 (8)].

LEMMA 5. Suppose that R satisfies (i) and (ii). Moreover, if R satisfies either (iii) or $(I)'_n$ then R is a subdirect sum of nil commutative and local rings.

PROOF. Obviously, if f is a homomorphism from R onto R^*, then $f(N) = N^*$ is the set of all nilpotent elements in R^* . Thus it can be easily seen that any subring and any homomorphic image of R satisfies (i) and (ii). Moreover, if R satisfies (iii) (resp. $(I)'_n$). then any subring and any homomorphic image of R satisfies (iii) (resp. $(I)'_n$). By Birkhoff's theorem R is isomorphic to a subdirect sum of subdirectly irreducible rings. Therefore we may (and shall) assume that R is a subdirectly irreducible ring. Now for each x in R, $x^m = x^{m+1}x'$ for some positive integer m and some x' in the subring generated by x. Thus $x^m x'^m = e$ is idempotent . Therefore if R satisfies either (iii) or $(I)'_n$, then by Lemma 2, e is a centeral element in R. But R is a subdirectly irreducible ring , hence $e=0$ or $e = 1 (if\ 1 \in R)$. On the other hand as it can be seen easily, $x^m = x^m e$; thus if R has no identity element it must be nil and therefore commutative, by (i). If R has an identity element then $e=0$ or $e=1$ implies that each $x \in R$ is either a nilpotent or a unit element in R, i.e. R is a local ring. This completes the proof .

LEMMA 6

Suppose that R satisfies (i) and (ii). Moreover if R satisfies $(I)_n$, then R is a subdirect sum of nil commutative and local rings.

PROOF. The proof is quite similar to that of Lemma5.

LEMMA 7. If N is a commutative ideal, then for all

STRUCTURE OF RINGS WITH CONSTRAINTS ON NILPOTENT ELEMENTS AND GENERALIZED COMMUTATORS

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ABSTRACT

Let R be an associative ring in which N the set of all nilpotent elements in R is commutative, and for each x in R there exists a polynomial f with integer coefficients such that $x - x^2 f(x) \in N$. *We discuss some of the conditions on R which imply that R is a subdirect sum of nil commutative and local commutative rings, obtaining some extensions of the results of [1].*

Throughout the present paper, R will represent an associative ring with center Z, and N the set of all nilpotent elements in R. Following [6], generalized commutators $[x,x_1,x_2,x_3,...,x_k]$ for positive integers k are defined as follows: $[x,x_1] = xx_1 - x_1x$ if $k=1$, and $[[x,x_1,x_2,x_3,...,x_{k-1}],x_k]$ if $k > 1$. For $x_1 = x_2 = x_3 = ... = x_k = y$, $[x,y,y,...,y]$ is abbereviated as $[x,y]_k$. We denote by Z(k)the set of all x in R such that $[x,x_1,x_2,x_3,...,x_k]=0$ for all $x_1,x_2,x_3,...,x_k$ in R; $Z(0) = 0$. A ringR is said to be *normal* if E the set of all idempotents in R is centeral.

The purpose of this paper is to prove the following generalization of the principle theorem of [1], which is also releated to a number of recent results by Tominaga and Yaqub (e.g.see [3], [4]).

Let n be a fixed positive integer, and consider the following conditions:

$(I)_n$ If $x,y \in R$ and $x-y \in N$, then either $x^n - y^n \in Z(\kappa)$ or both x,y are in $Z(k)$ for some positive integer $k=k(x,y)$ depending on x and y.

$(I)'_n$ If $x,y \in N$, then either $x^n - y^n \in Z(k)$ for some positive integer $k=k(x,y)$ depending on x and y, or both x,y commute with all nilpotent elements in R.

$(II)_m$ For any $a \in N$, $b \in R$ there exists a positive integer $k = k(a,b)$ such that $[(ab+b)^m - (ba+b)^m, b]_k = 0$ for some positive integer $m = m(a,b)$ with $(m,n) = 1$.

Q(n) For any $a \in N$, $b \in R$, $n[a,b] = 0$ implies that $[a,b] = 0$.

(i) N is commutative.

(ii) For each $x \in R$ there exists a polynomial f(x) with integer coefficients such that $x - x^2 f(x) \in N$.