CONCLUSION:

This paper has discussed the problem of separability for 2-D systems. Both transfer function and state space representation were considered. Several necessary and/or sufficient conditions for checking the separability of the 2-D system were stated. Presently we are investigating the partial separability problem of 2-D systems. This will be the subject of a future paper.

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when n=1 or m=1. For instance consider a 2-D system with dimension 7. We have the possibility of having the following cases

n+m	7	7	7	7	7	7
n	6	5	4	3	2	1
m	1	2	3	4	5	6
nm	6	10	12	12	10	6

In the first and last case we have six equations but in the third and forth case we have twelve equations.

We now state a necessary condition for separability.

Lemma (5):

If the system is separable then

$$\det(A) = \det(A_1)\det(A_4).$$

Proof:

From equation (3) we have $a_{nm} = a_{n0}a_{0m}$ where a_{nm}, a_{n0}, and a_{0m} are coefficients of $s_1^n s_2^m$, s_1^n, and s_2^m respectively. Thus

$a_{nm} = 1/d_{123\ldots(n+m)} = (-1)^{n+m}/\det(A)$

$a_n0 = d(n+1)\ldots(n+m)/d_{123}\ldots(n+m) = (-1)^n \det(A_4)/\det(A)$

$a_{0m} = d_{123}\ldots n/d_{123}\ldots(n+m) = (-1)^m \det(A_1)/\det(A)$

Substituting for a_{nm}, a_{n0}, and a_{0m} in equation (3) yields the result.

SPECIAL CASES

Case I (n=1, m=1)

The system is separable if and only if a_2 and/or a_3 are equal to zero.

Case II (n=1, m= arbitrary)

If the system is separable then $A_2A_3 = 0$.

Case I I I (m= 1, n= arbitrary)

If the system is separable then $A_3A_2 = 0$.

Example (2):

Given a 2-D system with the following state matrix

$$A=\begin{bmatrix} 5 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

The characteristic polynomial of above system is

$$Q(s_1, s_2) = (s_1 - 5)(s_2^2 - 4s_2 + 3).$$

It can easily be verified that $A_2A_3 = 0$. Note that A_2 and A3 are not equal to zero.

Complexity:

For calculating the coefficients of equation (6) we need to compute the determinants of all the minor matrices That is, we should calculate

$$\binom{n+m}{1}, \binom{n+m}{2}, \binom{n+m}{3}, \ldots\ldots\ldots, \binom{n+m}{n+m}$$

number of determinants of matrices with dimensions 1x1, 2x2, 3x3, ..., (n+m) x (n+m) respectively, where

$$\binom{i}{j} = \frac{i!}{j!\,(I-J)!}$$

It can easily be shown that if the matrix A_3 or A_2 is zero the equations (6a) are always satisfied. For instance for example (1) the equations (8) are

1) $\det(A) = \det(A_1)\det(A_4)$

2) $\det(A)\,[\mathrm{tr}(A_4)] = \det(A_4)\,[a_{33}\det(A_1) + a_{44}\det(A_1)]$

$= \det(A_4)\det(A_1)[\mathrm{tr}(A_4)]$

3) $\det(A)\,[\mathrm{tr}(A_1)] = \det(A_1)\,[a_{11}\det(A_4) + a_{22}\det(A_4)]$

$= \det(A_1)\det(A_4)\,[\mathrm{tr}(A_1)]$

4) $\det(A)\,[a_{11}a_{33} + a_{11}a_{44} + a_{22}a_{33} + a_{22}a_{44}] =$

$[a_{33}\det(A_1) + a_{44}\det(A_1)]\,[a_{11}\det(A_4) + a_{22}\det(A_4)]$

$= \det(A_1)\det(A_4)\,[a_{33}+a_{44}]\,[a_{11}+a_{22}]$

All above conditions are satisfied since $\det(A)=\det(A_1)\det(A_4)$

the rest of coefficients of equation (6). Note that . means determinant and A_{kk} is obtained from the matrix A by the change of its kth column with vector $[0 \;\; 0 \;\; ..... \; 0 \;\; 1 \;\; 0 .... 0]^T$ where one is in row k. The following example is stated for clarity of above procedure.

Example(1):

Consider a 2-D continuous system with matrix

$A = a_{ij} \qquad I = 1, 2, ...4 \qquad J = 1, 2, ...4$

where n= 2, and m= 2. Find the necessary and sufficient conditions for separability of above system.

Solution:

The characteristic polynomial is

$$Q(s_1, s_2) = q_{00} s_1^2 s^2 + q_{01} s_1^2 s2 + q_{10} s_1 s_2^2 + q_{02} s_1^2 + q_{11} s_1 s_2 + q_{20} s_2^2 + q_{12} s_1 + q_{21} s_2 + q_{22} \quad (7)$$

where

$$q_{2,2} = -A = (-1)^4 A = A = d_{1234}$$

$$q_{21} = (-1)^3 \left[\begin{bmatrix} a_{11} & a_{12} & 0 & a_{14} \\ a21 & a_{22} & 0 & a_{24} \\ a_{31} & a_{32} & 1 & a_{34} \\ a41 & a_{42} & 0 & a_{44} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & 1 \end{bmatrix} \right.$$

$$= (-1) \; [d_{124} + d_{123}]$$

$$q_{12} = (-1)^3 \left[\begin{bmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{bmatrix} + \begin{bmatrix} a_{11} & 0 & a_{13} & a_{14} \\ a_{21} & 1 & a23 & a_{24} \\ a_{31} & 0 & a_{33} & a34 \\ a_{41} & 0 & a_{43} & a_{44} \end{bmatrix} \right.$$

$$= (-1) \; [\, d_{234} + d_{134}]$$

$$q_{20} = (-1)^2 \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a3_1 & a_{32} & 1 & 0 \\ a_{41} & a_{42} & 0 & 1 \end{bmatrix} = d_{12}$$

$$q_{02} = (-1)^2 \begin{bmatrix} 1 & 0 & a_{13} & a_{14} \\ 0 & 1 & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix} = d_{34}$$

$$q_{11} =)-1)^2 \left[\begin{vmatrix} 1 & a_{12} & 0 & a_{14} \\ 0 & a_{22} & 0 & a_{24} \\ 0 & a_{32} & 1 & a_{34} \\ 0 & a_{42} & 0 & a_{44} \end{vmatrix} + \begin{vmatrix} 1 & a_{12} & a_{13} & 0 \\ 0 & a_{22} & a_{23} & 0 \\ 0 & a_{22} & a_{23} & 0 \\ 0 & a_{42} & a_{43} & 1 \end{vmatrix} + \begin{vmatrix} a11 & 0 & 0 & a_{14} \\ a21 & 1 & 0 & a_{24} \\ a21 & 0 & 1 & a_{24} \\ a41 & 0 & 0 & a_{44} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & a_{13} & 0 \\ a_{21} & 1 & a_{23} & 0 \\ a_{21} & 0 & a_{23} & 0 \\ a_{41} & 0 & a_{43} & 1 \end{vmatrix} \right]$$

$$= [d_{24} + d_{23} + d_{14} + d_{13}]$$

$$q0_1 = (-1) \left[\begin{vmatrix} 1 & 0 & 0 & a_{14} \\ 0 & 1 & 0 & a_{24} \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & a_{44} \end{vmatrix} + \begin{vmatrix} 1 & 0 & a_{13} & 0 \\ 0 & 1 & a_{23} & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & a_{43} & 1 \end{vmatrix} \right]$$

$$= (-1) \; [d_4 + d_3]$$

$$q_{10} = (-1) \left[\begin{vmatrix} a_{11} & 0 & 0 & 0 \\ a21 & 1 & 0 & 0 \\ a31 & 0 & 1 & 0 \\ a_{41} & 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & a_{12} & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & a_{32} & 1 & 0 \\ 0 & a_{42} & 0 & 1 \end{vmatrix} \right]$$

$$= (-1) \; [d_1 + d_2]$$

$$q_{00} = (+1)^0 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1$$

We now divide the equation (7) by d_{1234} in order to get exactly the same form as equation (2a). Applying the condition (3) to the polynomial (7) implies that the following condition are needed to be satisfied for separability.

1) $d_{1234} = d_{12} \cdot d_{34}$
2) $d_{1234} (d_3 + d_4) = d_{34} (d_{123} + d_{124})$
3) $d_{1234} (d_1 + d_2) = d_{12} (d_{134} + d_{234})$
4) $d_{1234} (d_{13} + d_{14} + d_{23} + d_{24}) = (d_{123} + d_{124}) (d_{134} + d_{234})$

(8)

Remark 1.

It should be noted that the number of equations that need to be satisfied for separability are equal to nm. Note that $nm = [(n + m)^2 - (n - m)] \; 1/4$. Thus assuming (n +m) is fixed the number of equations are minimum if (n - m) is maximum. This comes posible

PROPERTIES OF MATRIX D

If the system of equation (1) is separable, then the matrix D defined in equation (3a) has the following properties

1) rank (D) = 1

2) assuming n = m, the matrix D has the following eigenvalues

$$o_i = \begin{cases} 0 & i = 0,1,2 \dots n \\ tr(D) = \sum_{i=0}^{n} a_{ii} & i = n+1 \end{cases}$$

3) assuming n = m, the matrix D is nilpotent if tR(D) = 0.

Note that if the terms $(s_1 s_2)^i$ are missing in $Q(s_1, s_2)$ for all i, then tr (D) =0.

Above lemma can be generalized for m – D systems.

CASE II

Now we consider the linear, time invariant countinuous – time 2 – D systems described by porter [9].

$$\begin{bmatrix} \phi_x(X,Y) \\ \phi_y(X,Y) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} \phi(X,Y) \\ \phi(X,Y) \end{bmatrix} + \begin{bmatrix} B_1 & \\ & B_2 \end{bmatrix} U(X,Y)$$

$$r(X,Y) = [C_1 C_2] \begin{bmatrix} \phi(X,Y) \\ \phi(X,Y) \end{bmatrix} \qquad (5)$$

where $\Phi \in R^n, \phi \in R^m, u \in R^P$ is the input vector, $r \in R^q$ is the output vector and A, B, C are constant matrices of appropriate dimensions.

We are intezested to know the conditions on the state matrix A that guarantees the separability of the system presented by equation (5). It is a well known result [10] that if the submatrices A_2 and /or A_3 are zero then the corresponding transfer function has a separable denominator. This is because the characteristic polynomial of the system is equal to multiplication of characteristic polynomial of each subsystem. However we could have separable systems with A_2 and/or A_3 being non zero. Therefore above conditions are only sufficient for separability. The best choice for obtaining the necessary and sufficient conditions for separability of a 2 – D system is that we calculate the characteristic polynomial of the system and then use the result of lemma (2). In the following we show how to compute the characteristic polynomial.

Applying the 2 – D Laplace transform to equation (5), we obtain the following 2 – D transfer function.

$$H(s_1, s_2) = C\,[SI - A]^{-1} B$$

Where

$$S = \begin{bmatrix} S1 - I_n & 0 \\ 0 & s_2 - I_m \end{bmatrix} = s_1 I_n \oplus s_2 I_m$$

and + denotes the direct sum of matrices.

The characteristic polynomial of system equation (5) is equal to det (si – A). This may be written in the form

$$Q(s_1, s_2) = \sum_{i=0}^{n} \sum_{j=0}^{m} q_{ij} s_1^i s_2^j \quad \text{with } q_{00} = 1$$

where the coefficient qij can be calculated as follows:

$$q_{n,m} = \det[sI - A]_{s1=0,\, s2=0} = |A| = (-1)^{n+m} |A|$$

$$q_{n-1,m} = \frac{\delta}{\delta s_1} \det[sI - A]_{s1=0,\, s_2=0} = (-1)^{n+m-1} \sum_{k=1}^{n} |A_{kk}|$$

$$q_{n,m-1} = \frac{\delta}{\delta s_2} \det[sI - A]_{s_1=0,\, s_2=0} = (-1)^{n+m-1} \sum_{k=n+1}^{n+m} |AKK| \qquad (6a)$$

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By taking second $(\delta^2/\delta s_2 \delta s_1,\ \delta^2 s_1^2,\ \delta^2/\delta s_2^2)$, third, ... nth partial derevitives we can calculate the separability.

Lemma(1): Considering equation (2), the system of equation (1) is separable if and only if the polynomials $a_i(s_2)$ have same roots for all i= 1,2,..... n.

Proof:

Necessary part

The polynomials $a_i(s_2)$ having the same roots, implies that $k_0a_0(s_2) = k_1a_1(s_2) = k_2a_2(s_2) = \dots\dots\dots\dots = k_na_n(s_2) = Q_2(s_2)$ for some constant k_i. Substituting for $a_i(s_2)$ in terms of $Q_2(s_2)$ in equation (2) yields the following:

$$Q(S_1, S_2) = \sum_{i=0}^{n} (1 / K_i)\, Q_2(S_2)\, S_1^{\,i}$$
$$= Q_2(S_2) \sum_{i=0}^{n} (1/Ki)\, S_1^{\,i}$$
$$= Q_2(S_2)\, Q_1(S_1)$$

SUFFICIENT PART

Assuming the system is separable, that is

$$Q(s_1, s_2) = Q_2(s_2)\, Q_1(s_1)$$
$$= Q_2(s_2)\, (c_0 + c_1s_1 + \dots\dots c_ms_1^{\,m})$$
$$= Q_2(s_2)c_0 + Q_2(s_2)\, c_1s_1 + \dots\dots Q_2(s_2)c_ms_1^{\,m}$$
$$= \sum_{i=0}^{n} C_iQ_2(S_2)\, S_1^{\,i}$$
$$= \sum_{i=0}^{n} a_i(S_2)\, S_1^{\,i}$$

Where $a_i(s_2) = c_iQ_2(s_2)$.

Therefore all $a_i(s_2)$ have the same roots as $Q_2(s_2)$.

Note that it is possible, even if the system is not separable, the polynomials $a_i(s_2)$ have some roots in common, namely partialy separable. An algorithm for calculating these roots are going to be the subject of a future paper.

Lemma(2):

Considering equation (2a) with $a_{00}=1$, the system of equation (1) is separable if and only if

$$a_{ij} = a_{i0}\, a_{0j} \qquad \text{for} \quad i = 1,2,\dots. n \qquad j= 1,2,\dots\dots m \tag{3}$$

Proof: The polynomial $Q(s_1, s_2)$ can be written in the form

$$Q(s_1, s_2) = [1\ s_2 \dots s_2m]\quad D\ [1\ s_1 \dots s_1n]^T \tag{3a}$$

Where

$$D = \begin{bmatrix} a_{00} & a_{10} & a_{20} \dots\dots\dots & a_{n0} \\ a_{01} & a_{11} & \dots\dots\dots\dots\dots & a_{n1} \\ & & & \\ a_{0m} & a_{1m} & \dots\dots\dots\dots\dots & a_{nm} \end{bmatrix}$$

If the condition (3) is satisfied the matrix D has maximum rank one and can be decomposed as

$$D = [1\ a_{01} \dots\dots a_{0m}]^T\ [1\ a_{10} \dots\dots a_{n0}]$$

Thus equation (3a) becomes

$$Q(s_1, s_2) = (1+a_{01}s_2 + \dots + a_{0m}s_2^{\,m})\ (1+a_{10}s_1 + \dots + \dots + a_{n0}s_1^{\,n}) \tag{4}$$

Therefore $Q(s_1, s_2)$ is separable. The sufficient part can be easily shown by multiplying out the equation (4) and matching the coefficient with equation (2a).

Lemma (3):

If each row of the matrix D is multiple of first row by a constant then $Q(s_1, s_2)$ is separable.

PROOF:

Above assumption implies that the matrix D can be decomposed as a multiplication of two vectors. That is $D = \underline{k}.\ \underline{a}$ where $\underline{k} = [1\ k_1\ \ k_2 \dots.. k_m]^T$, $\underline{a} = [1\ \ a_{10}\ a_{20} \dots.. a_{n0}]$. Thus $Q(s_1, s_2) = (1 + k_1\, s_2 + \dots.. k_ms_2^{\,m})\,(1+ a_{10}\, s_1 + \dots.. a_{n0}s_1^{\,n})$.

LEMMA (4): IF

$$a_{ij} \begin{cases} = 0 & i = j = 0 \\ \neq 0 & \text{Otherwise} \end{cases}$$

then $Q(s_1, s_2)$ defined in equation (2a) can not be factored to two 1 – D polynomials for any arbitrary n and m.

PROOF:

The proof follows from lemma (2).

Separability of 2 – D System

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ABSTRACT

This paper presents several tests for checking the separability of a given two variable (2– V) polynomial. The 2 – V polynomial can be considered as denominator of a transfer function representation of a 2 – D system. The separability condition is stated in terms of the coefficient of the 2 – V polynomials. This can be extended to m – V polynomial. Considering the 2 – D state space model representation with the state matrix A, we obtain the necessary and sufficient conditions on the matrix A that gurantees the separability of the 2 – D system. We also show that the number of equations that need to be satisfied for separability are minimum if and only if n = 1 and or m = 1.

NTRODUCTION

There has been considerable interest in recent 'ears in the design of two–dimensional (2–D) systems. nsuring the stability [1–8] of the 2 – D system is one ›f the major problems. Due to the difficulty of testing he stability of an arbitrary 2 – D system it is preferable o work with the separable systems. A 2 – D system is :alled separable if its characteristic polynomial (den–noinator) can be factored to two 1 – D polynomials i. e. $Q(s_1, s_2) = Q_1(s_1)\, Q_2(s_2)$). In this case the sta–›ility test becomes straight forward. Therefore it is lesirable to know if the system is separable or not. We vill state algorithms for determining the separability of ı continuoustime 2 – D system. First we consider the ransfer function representation. Next the state space representation is cosidered.

Case I

Consider a 2 – D analog filter

$$H(s_1, s_2) = \frac{P(s_1, s_2)}{Q(s_1, s_2)} \qquad (1)$$

The polynomial $Q(s_1, s_2)$ can be written in the following forms:

$$Q(S_1, S_2) = \sum_{i=0}^{n} a_i(S_2)\, S_1^{\,i} \qquad (2)$$

$$Q(S_1, S_2) = \sum_{j=0}^{m} a_{ij}\, S_1^{\,i}\, S_2^{\,j} \qquad (2a)$$

Where $a_i(s_2)$ are polynomials interms of s_2 and a_{ij} are constant.

We now state the following lemmas for determinig